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Finite rigid subgraphs of the pants graphs of punctured spheres

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ABSTRACT

We prove a strong form of finite rigidity for pants graphs of spheres. Specifically, for any $n \geq 4$, we construct a finite subgraph X_n of the pants graph $\mathcal{P}(S_{0,n})$ of the n-punctured sphere $S_{0,n}$ with the following property. Any simplicial embedding of X_n into any pants graph $\mathcal{P}(S_{0,m})$ of a punctured sphere is induced by an embedding $S_{0,n} \to S_{0,m}$.

1. Introduction

Let $S = S_{g,n}$ be an orientable surface of genus g with n punctures. The pants graph $\mathcal{P}(S)$ of S has vertices corresponding to pants decompositions of S and edges corresponding to elementary moves (see Section 2 for details).

Margalit [8] proved that, for most surfaces S, $\text{Aut}(\mathcal{P}(S)) \cong \text{Mod}^{\pm}(S)$, where $\text{Mod}^{\pm}(S) = \pi_0(\text{Homeo}(S))$ is the extended mapping class group. This result was extended by Aramayona [1] who proved that, for any two surfaces S and S' such that the complexity of S is at least 2, every injective simplicial map $\phi : \mathcal{P}(S) \to \mathcal{P}(S')$ is induced by a π_1-injective embedding $f : S \to S'$. For related results on curve complexes see [5–7,10,2].

$\mathcal{P}(S)$ is an infinite and locally infinite graph. In this paper, we refine Aramayona's result (for punctured spheres) and prove the following.

Theorem 1.1. *For $n \geq 4$, there exists a finite subgraph $X_n \subset \mathcal{P}(S_{0,n})$ such that for any punctured sphere $S_{0,m}$ and any injective simplicial map*

$$\phi : X_n \to \mathcal{P}(S_{0,m}),$$

there exists a π_1-injective embedding $f : S_{0,n} \to S_{0,m}$ that induces ϕ.

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For $n = 4$, the isotopy class of f is unique up to precomposing by an element $\sigma \in \mathrm{Mod}(S_{0,4})$ inducing the identity on $\mathcal{P}(S_{0,4})$.

For $n \geq 5$, the isotopy class of f is unique.

We say that f **induces** ϕ if there is a deficiency-$(n-3)$ multicurve Q on $S_{0,m}$ with the following property (see Section 2 for definitions). The image $f(S_{0,n})$ is the unique non-pants component $(S_{0,m} - Q)_0 \subset S_{0,m} - Q$ and the simplicial map

$$f^Q : \mathcal{P}(S_{0,n}) \to \mathcal{P}(S_{0,m}),$$

defined by $f^Q(u) = f(u) \cup Q$ satisfies $f^Q(u) = \phi(u)$ for any vertex $u \in X_n$.

This result is analogous to Aramayona–Leininger [2] for the case of the curve complex when $n = m$ (that theorem applies to arbitrary surfaces).

One of the main difficulties in proving Theorem 1.1 is to construct the finite subgraphs X_n. To do this we can look for a candidate subgraph which, under additional hypotheses on the simplicial map, allows us to construct the embedding of the surface. For example, we have the condition on the simplicial map of Z_5 in Lemma 2.3. We then need to enlarge the candidate subgraph so that those extra conditions are encoded in the enlarged subgraph. But then another problem might arise which is that the induced map of the embedding that works on the original candidate subgraph might not control the added parts in the enlarged subgraph. For example, a simplicial embedding of the thick pentagon $\widehat{Z_5}$ ensures that the simplicial map restricted to Z_5 satisfies Lemma 2.3 and hence there is a candidate embedding of $S_{0,5}$. But the induced map may not agree with the simplicial map on $\widehat{Z_5} - Z_5$. Then we have to enlarge the subgraph further which might cause more problems.

It seems likely that Theorem 1.1 should be true for essentially any surface S. However it is unclear how to choose a subgraph $X \subset P(S)$.

Outline of the paper. Section 2 contains the relevant background and definitions. This section also contains the proof of the theorem in the case of $n = 4$. In Section 3, we describe the finite subgraph X_n and prove some important properties of the subgraph. We prove the theorem for the case of $n = 5$ in Section 4 and for the general case in Section 5.

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2. Background and definitions

Here we describe some of the background material. See [1] and [8] for more details. Let $S = S_{g,n}$ be an orientable surface of genus g with n punctures. A simple closed curve on S is **essential** if it does not bound a disk or a once-punctured disk on S. In this paper, a **curve** is a homotopy class of essential simple closed curves on S.

Let γ and γ' be curves on S. The **geometric intersection number** of γ and γ' is defined as the minimum number of transverse intersection points among the simple representatives of γ and γ'.

The intersection of any two curves mentioned in this paper refers to their geometric intersection number. Whenever we represent homotopy classes γ and γ' by simple closed curves, we assume these intersect in $i(\gamma, \gamma')$ points and we will not distinguish a homotopy class from its representatives. Two curves γ and γ' are **disjoint** if $i(\gamma, \gamma') = 0$. Let A be a set of curves on S. We say that $\bigcup_{\alpha \in A} \alpha$ **fills** S if the complement is a disjoint union of disks or once-punctured disks.

We call a surface which is homeomorphic to $S_{0,3}$, **a pair of pants**. Let A be a set of pairwise disjoint curves on S. The **nontrivial component(s)** of the complement of the curves in A, denoted $(S - A)_0$, is the union of the non-pants components of the complement.