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Ideals and idempotents in the uniform ultrafilters

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ABSTRACT

For every discrete semigroup S, the space βS (the Stone–Čech compactification of S) has a natural, right-topological semigroup structure extending S. Under some mild conditions, $U(S)$, the set of uniform ultrafilters on S, is a two-sided ideal of βS, and therefore contains all of its minimal left ideals and minimal idempotents. Our main theorem states that, if S satisfies some mild distributivity conditions, $U(S)$ contains prime minimal left ideals and left-maximal idempotents.

If S is countable, then $U(S) = S^*$, and a special case of our main theorem is that if a countable discrete semigroup S is weakly cancellative and left-cancellative, then $S^* = \beta S \setminus S$ contains prime minimal left ideals and left-maximal idempotents. We will provide examples of weakly cancellative semigroups where these conclusions fail, thus showing that this result is fairly sharp.

1. Introduction

Semigroups of the form βS and S^* have played a prominent part in modern combinatorics and algebra. Particularly important have been the minimal left ideals and the minimal idempotents, and these have become the object of a good deal of justifiable curiosity. Our main theorem concerns the existence of minimal left ideals and minimal idempotents with special topological and algebraic properties:

Main Theorem (abridged). *Let S be a countable discrete semigroup. If S is weakly cancellative, then*

(1) *there is a minimal left ideal L of S^* that is also a weak P-set.*

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If S is also left-cancellative, then

(2) *L is prime in S^*;*
(3) *the idempotents in L are left-maximal in S^* (in particular, these idempotents are simultaneously minimal and maximal).*

In the special case that S is a countable group, (2) and (3) were proved by Zelenyuk in [11]. Before the appearance of Zelenyuk's paper, the existence of left-maximal idempotents in S^* was a longstanding open problem. Even for $S = (\mathbb{Z}, +)$, this problem went unresolved for years, despite the fact that a good deal of work was done on and around the problem (see Questions 9.25 and 9.26 in [6]; see also Questions 5.5(2), (3) in [7], Problems 4.6 and 4.7 in [4], and [10]). The present proof uses a different technique from Zelenyuk's. A special case of the present proof, when $S = (\mathbb{N}, +)$, was given by the author and Jonathan Verner in [5].

This "abridged" version of our main theorem solves the problem of left-maximal idempotents in S^* in a very general setting. We will show by example that left cancellativity is necessary to prove (2) and (3), so that the result is fairly sharp.

The unabridged version of our main theorem concerns not S^* but $U(S)$, the set of uniform ultrafilters on S. If S is countable, then $U(S) = S^*$, but in the uncountable setting this is not the case. However, assuming S is very weakly cancellative (defined in the next section), $U(S)$ is a two-sided ideal (hence a subsemigroup) of βS. In particular, the minimal left ideals of $U(S)$ are precisely the minimal left ideals of S^* and βS.

Main Theorem (unabridged). *Let S be a discrete semigroup with $|S| = \kappa$, where κ is regular. If S is very weakly cancellative, then*

(1) *there is a minimal left ideal L that is a weak P_{κ^+}-set in $U(S)$.*

If S is also left-cancellative, then

(2) *L is prime in $U(S)$;*
(3) *the idempotents in L are left-maximal in $U(S)$ (in particular, these idempotents are simultaneously minimal and maximal).*

In Section 2, we provide some background material concerning semifilters, ultrafilters, and the semigroups βS, S^*, and $U(S)$ (we also provide definitions of all the terms appearing in this introduction). In Section 3, we prove the abridged version of our main theorem and provide examples showing that the result is sharp. We have chosen to prove the abridged version first to highlight the main pattern of the proof, which is the same for the unabridged version. The proof of the abridged version hinges on a result from [5] concerning filters and semifilters on countable sets. Before proving the unabridged version, we need to extend this result to uncountable sets of regular cardinality, which is done in Section 4 by building on the work of Baker and Kunen in [1]. In Section 5, we prove the unabridged version of our main theorem.

2. Definitions and preliminaries

2.1. Filters, semifilters, and ultrafilters

A *semifilter* on a set S is a subset $\mathcal{G}$ of $\mathcal{P}(S)$ such that

- (nontriviality) $\emptyset \neq \mathcal{G} \neq \mathcal{P}(S)$;