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Metric spaces with complexity of the smallest infinite ordinal number ☆

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A B S T R A C T

In this paper, we are concerned with the study of metric spaces with complexity of the smallest infinite ordinal number. We give equivalent formulations of the definition of metric spaces with complexity of the smallest infinite ordinal number and prove that the exact complexity of the finite product $\mathbb{Z} \wr \mathbb{Z} \times \mathbb{Z} \wr \mathbb{Z} \times \cdots \times \mathbb{Z} \wr \mathbb{Z}$ of wreath products is ω, where ω is the smallest infinite ordinal number. Consequently, we obtain that the complexity of $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ is $\omega + 1$.

1. Introduction

Inspired by the property of finite asymptotic dimension of M. Gromov ([1]), a geometric concept of finite decomposition complexity was introduced by E. Guentner, R. Tessera and G. Yu. Roughly speaking, a metric space has finite decomposition complexity when there is an algorithm to decompose the space into nice pieces in certain asymptotic way. It turned out that many groups have finite decomposition complexity and these groups satisfy strong rigidity properties including the stable Borel conjecture ([2], [3]). In [3], E. Guentner, R. Tessera and G. Yu show that the class of groups with finite decomposition complexity includes all linear groups, subgroups of almost connected Lie groups, hyperbolic groups and elementary amenable groups and is closed under taking subgroups, extensions, free amalgamated products, HNN-extensions and inductive limits.

Finite decomposition complexity is a large scale property of a metric space. To make the property quantitative, a countable ordinal "the complexity" can be defined for a metric space with finite decomposition complexity. There is a sequence of subgroups of Thompson's group F which is defined by induction as follows:

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$$G_1 = \mathbb{Z} \wr \mathbb{Z}, \quad G_{n+1} = G_n \wr \mathbb{Z}.$$

We are concerned with the study of the exact complexity of G_n which is partially inspired by the question of the finite decomposition complexity of Thompson's group F ([4], [5], [6]). In fact, if the exact complexity of the sequence $\{G_n\}$ of subgroups of F is strictly increasing, then we can prove that Thompson's group F does not have finite decomposition complexity. In [6], we proved that the complexity of G_n is ωn and the exact complexity of $\mathbb{Z} \wr \mathbb{Z}$ is ω, where ω is the smallest infinite ordinal number, but it is still unknown about the exact complexity of G_n when $n > 1$. Here we prove that the exact complexity of the finite product $\mathbb{Z} \wr \mathbb{Z} \times \mathbb{Z} \wr \mathbb{Z} \times \cdots \times \mathbb{Z} \wr \mathbb{Z}$ of wreath products is ω. Consequently, we obtain that the complexity of $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}$ is $\omega + 1$.

There is no group of examples known which demonstrate a difference between the exact complexity of ω and the exact complexity of α, where α is a countable ordinal greater than ω. So the question arises naturally: Is there any metric space with the exact complexity greater than ω? Here we give definitions of metric spaces with complexity of ω.

2. Equivalent descriptions of metric spaces with complexity of ω

We begin by recalling some elementary concepts from coarse geometry.

Let (X, d) be a metric space. For $U, V \subseteq X$, let

$$\text{diam } U = \sup\{d(x, y) : x, y \in U\}$$

and

$$d(U, V) = \inf\{d(x, y) : x \in U, y \in V\}.$$

A family $\mathcal{U}$ of subsets of X is said to be *uniformly bounded* if *diam* $\mathcal{U} \triangleq \sup\{\text{diam } U : U \in \mathcal{U}\}$ is finite.

A family $\mathcal{U}$ of subsets of X is said to be *r-disjoint* if

$$d(U, V) \geq r \qquad \text{for every } U \neq V \in \mathcal{U}.$$

Definition 2.1. ([7]) A metric space X has *finite asymptotic dimension* if there is a $n \in \mathbb{N}$, such that for every $r > 0$, there exists a sequence of uniformly bounded families $\{\mathcal{U}_i\}_{i=1}^n$ of subsets of X such that the union $\bigcup_{i=1}^n \mathcal{U}_i$ covers X and each $\mathcal{U}_i$ is r-disjoint.

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families. A map of families from $\mathcal{X}$ to $\mathcal{Y}$ is a collection of functions $F = \{f\}$, each mapping some $X \in \mathcal{X}$ to some $Y \in \mathcal{Y}$ and such that every $X \in \mathcal{X}$ is the domain of at least one $f \in F$. We use the notation $F : \mathcal{X} \to \mathcal{Y}$ and, when confusion could occur, write $f : X_f \to Y_f$ to refer to an individual function in F.

Definition 2.2. A map of families $F : \mathcal{X} \to \mathcal{Y}$ is *uniformly expansive* if there exists a non-decreasing function $\theta : [0, \infty) \to [0, \infty)$ such that for every $f \in F$ and every $x, y \in X_f$,

$$d(f(x), f(y)) \leq \theta(d(x, y)).$$

$F : \mathcal{X} \to \mathcal{Y}$ is *effectively proper* if there exists a proper non-decreasing function $\delta : [0, \infty) \to [0, \infty)$ such that for every $f \in F$ and every $x, y \in X_f$,

$$d(f(x), f(y)) \geq \delta(d(x, y)).$$

And $F : \mathcal{X} \to \mathcal{Y}$ is a *coarse embedding* if it is both uniformly expansive and effectively proper.