Topology and its Applications 236 (2018) 11–25

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Different types of relative contractibility and their applications

Mirosław Ślosarski

Koszalin University of Technology, Śniadeckich 2, PL-75-453 Koszalin, Poland

ARTICLE INFO

Article history:
Received 12 September 2017
Accepted 13 December 2017
Available online 27 December 2017

MSC:
54D05
54C60
54C56
54C15

Keywords:
Movable space
Cell-like map
Movable map
$\mathbb{D}$-contractibility
$\mathbb{CE}$-contractibility
$\mathbb{MOV}$-contractibility
Absolute relative retract
Absolute neighborhood relative retract

ABSTRACT

In this article the new properties of relative retracts in the context of relative homotopy are studied. The results of the studies are particularly applied to the characterization of connected and locally connected spaces and absolute neighborhood retracts.

1. Introduction

A few articles have been written that contain the results of studies of relative retracts and a relative homotopy as well as their applications to fixed point theory, the theory of coincidence and the characterization of metrizable spaces. In particular the article [10,11] is recommended. In my opinion the theory of relative retracts (see, [11]) and the theory of a relative homotopy (see, [10]) have interesting properties and many various applications. The article in paragraph 3 presents a class of relative retracts (in regard to movable maps (see, Theorem 3.11)) that overlaps the class of compact, connected and locally connected spaces. It means that the properties of these spaces can be studied with the use of the properties of relative retract and a relative homotopy. In paragraph 4 a class of relative retracts (in regard to cell-like maps) is recalled. It is shown (see Proposition 3.6 and Proposition 4.3) that in the context of new topological tools

E-mail address: slosmiro@gmail.com.

https://doi.org/10.1016/j.topol.2017.12.024
0166-8641/

(**MOV**-contractibility, strongly locally $\mathbb{CE}$-contractibility) they have similar properties to the properties of retracts in the sense of Borsuk (see, [1]).

2. Preliminaries

Throughout this paper all topological spaces are assumed to be metrizable. A continuous mapping $f : X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is nonempty and compact. Let X and Y be two spaces and assume that for every $x \in X$ a nonempty and compact subset $\varphi(x)$ of Y is given. In such a case, we say that $\varphi : X \multimap Y$ is a multivalued map. For a multivalued map $\varphi : X \multimap Y$ and a subset $A \subset Y$, we let:

$$\varphi^{-1}(A) = \{x \in X;\ \varphi(x) \subset A\}.$$

A multivalued map $\varphi : X \multimap Y$ is called upper semicontinuous (u.s.c.) provided for every open $U \subset Y$ the set $\varphi^{-1}(U)$ is open in X. Let H_* be the *Čech* homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of a graded vector space and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being a q-dimensional *Čech* homology group with compact carriers of X. For a continuous map $f : X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$ where $f_q : H_q(X) \to H_q(Y)$ ([5]). A space X is acyclic if:

(i) X is non-empty,
(ii) $H_q(X) = 0$ for every $q \geq 1$ and
(iii) $H_0(X) \approx \mathbb{Q}$.

A proper map $p : X \to Y$ is called Vietoris provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic. We recall that if $p : X \to Y$ is a Vietoris map then

$$p_* : H_*(X) \to H_*(Y) \tag{1}$$

is an isomorphism. A proper map $p : X \to Y$ is called cell-like provided for each $y \in Y$ the set $p^{-1}(y)$ has a trivial shape (in the sense of Borsuk (see [2])). We know that a compact set of trivial shape is acyclic. Hence, if $p : X \to Y$ is a cell-like map, then it is a Vietoris map.

Definition 2.1. Let X be an ANR and let $X_0 \subset X$ be a closed subset. We say that X_0 is movable in X provided every neighborhood U of X_0 admits a neighborhood U' of X_0, $U' \subset U$, such that for every neighborhood U'' of X_0, $U'' \subset U$, there exists a homotopy $H\colon U' \times [0,1] \to U$ with $H(x,0) = x$ and $H(x,1) \in U''$, for any $x \in U'$.

Definition 2.2. Let X be a compact metrizable space. We say that X is movable provided there exists $Z \in ANR$ and an embedding $e : X \to Z$ such that $e(X)$ is movable in Z.

Let us notice that the property of being movable is an absolute property, that is if A is a movable set in some ANR X and $j : A \to X'$ is an embedding into an ANR X', then $j(A)$ is movable in X' (see [2]). A proper map $p : X \to Y$ is called movable provided for each $y \in Y$ the set $p^{-1}(y)$ is movable. A compact set of trivial shape is movable (see [2]). Hence, if $p : X \to Y$ is a cell-like map, then it is a movable map.

Proposition 2.3. *[2] Let X and Y be compact spaces. The space $X \times Y$ is movable if and only if X and Y are movable spaces.*