Topology and its Applications 236 (2018) 26–43

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Berge duals and universally tight contact structures

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ARTICLE INFO

Article history:
Received 27 June 2017
Received in revised form 29 December 2017
Accepted 1 January 2018
Available online 4 January 2018

Keywords:
Low-dimensional topology
Knot theory
Contact geometry
Dehn surgery
Open book decompositions
Lens spaces

ABSTRACT

Dehn surgery on a knot determines a *dual* knot in the surgered manifold, the core of the filling torus. We consider duals of knots in S^3 that have a lens space surgery. Each dual supports a contact structure. We show that if a universally tight contact structure is supported, then the dual is in the same homology class as the dual of a torus knot.

1. Introduction

Let p, q be coprime with $0 \leq q < p$. Considering S^3 as the unit sphere of $\mathbb{C}^2$, with coordinates (z_1, z_2), define the lens space $L(p, q)$ as the quotient of S^3 under the $(\mathbb{Z}/p\mathbb{Z})$-action $(z_1, z_2) \sim (\omega z_1, \omega^q z_2)$ where $\omega = \exp(2\pi\sqrt{-1}/p)$. Write $\pi : S^3 \to L(p, q)$ for the associated covering map.

If there is an integer surgery on a knot $K \subset S^3$ that produces a lens space $L(p, q)$ then we call K a *lens space knot*.[1] Passing to the mirror of K if needed, we consider only positive surgeries. A method of constructing lens space knots was introduced in [4]. Let Σ be a standard genus 2 surface splitting $S^3 = H_1 \cup_\Sigma H_2$ as two handlebodies. Any simple closed curve in Σ that is primitive in each free group $\pi_1(H_i)$, $i = 1, 2$ (under the inclusion $\Sigma \subset H_i$), has a lens space surgery with framing given by Σ. Knots represented by such a simple closed curve are called *doubly primitive*. The Berge Conjecture [20, Problem 1.78] is that every lens space knot is doubly primitive.

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[1] While this terminology is common, it does sound as though it were describing a knot in a lens space. We will try to minimize the confusion by often referring to the ambient manifold.

https://doi.org/10.1016/j.topol.2018.01.001
0166-8641/

Berge catalogued ten families of doubly primitive knots which we refer to as Types I through X.[2] Following convention we call the knots in Types I through X the *Berge knots*, though it is now known that every doubly primitive knot is one of the Berge knots [14].

Recall that Dehn surgery on $K \subset S^3$ picks out a knot in the surgered manifold, the *surgery dual*, defined as the core of the surgery solid torus. As elsewhere in the literature, if $K \subset S^3$ is a Berge knot and $K' \subset L(p,q)$ its surgery dual, we also call $K' \subset L(p,q)$ a Berge knot. A number of investigations take this dual perspective (e.g. [8], [14], [16], [27]).

By the work of numerous authors, a dual in $L(p,q)$ to a lens space knot in S^3 is rationally fibered and supports a tight contact structure on $L(p,q)$ (see Section 2). A universally tight contact structure on $L(p,q)$ is obtained by using π to push forward the standard contact structure on S^3. Denote the resulting contact structure on $L(p,q)$ by $\xi_{p,q}$. When $0 < q < p-1$ there is another universally tight contact structure, obtained by reversing the coorientation of the contact planes. We consider which duals to a lens space knot support one of these contact structures.

Theorem 1.1. *If $K \subset L(p,q)$ is dual to a lens space knot and K supports a universally tight contact structure, then the homology class of K in $H_1(L(p,q))$ contains a Berge knot that is dual to a torus knot in S^3.*

Remark 1.2. Here, and throughout this paper, homology is taken with integer coefficients.

Remark 1.3. We will prove (see Theorem 4.1) that a Berge knot $B \subset L(p,q)$ that supports a universally tight structure is dual to a torus knot. However, our techniques use only the homology class of B and its *rational* genus (see Section 2). That the homology class of a lens space knot K contains a Berge knot which has the same knot Floer homology as K, and thus genus, was shown in [14] (see also [27, Theorem 2]).

Remark 1.4. Theorem 1.1 implies, for a dual K to a lens space knot, that the natural extension of the transverse invariant λ^+ of [26] to the setting of lens spaces does not live in the extremal Alexander grading of $\widehat{HFK}(L(p,q),K)$ unless K is homologous to a torus knot dual.

Theorem 1.1 has a consequence for fractional Dehn twist coefficients of open books (see [22]) which are supported by lens space knots.

Corollary 1.5. *If $K \subset S^3$ is a lens space knot with a surgery dual that is not homologous to the dual of a torus knot, then $c(h) < \frac{2}{2g(K)-1}$, where $g(K)$ is the Seifert genus, h is the monodromy for K, and $c(h)$ the fractional Dehn twist coefficient.*

We should remark that the bound of Corollary 1.5 (in fact, a slightly better one) can be obtained by using [22, Theorem 4.5] in place of Theorem 1.1 in the proof (see Section 4).

In Section 2 we review work of Baker, Etnyre, and Van Horn-Morris, which allows us to approach Theorem 1.1 through the rational Bennequin–Eliashberg inequality. We also review how to compute the self-linking number of transverse knots in $\xi_{p,q}$ and what is known about homology classes of the surgery duals in $H_1(L(p,q))$. Section 3 addresses the sharpness of the rational Bennequin–Eliashberg inequality for Berge knots in each of the different types. The proofs of Theorem 1.1 and Corollary 1.5 are given in Section 4.

[2] In fact, Berge described twelve families, I–XII. As in [27], our description in Types IX and X account for those in XI and XII by allowing the parameter j to be negative.