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Biquasile colorings of oriented surface-links

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ABSTRACT

We introduce colorings of oriented surface-links by biquasiles using marked graph diagrams. We use these colorings to define counting invariants and Boltzmann enhancements of the biquasile counting invariants for oriented surface-links. We provide examples to show that the invariants can distinguish both closed surface-links and cobordisms and are sensitive to orientation.

1. Introduction

In [9], a type of algebraic structure known as *biquasiles* was introduced and used to define invariants of oriented classical knots and links via counting colorings of graphs known as *dual graph diagrams*. Dual graph versions of a generating set of oriented Reidemeister moves were identified and used to motivate the biquasile axioms and to prove invariance of the set of biquasile colorings under these moves. Biquasiles can be understood as a special case of the ternary algebraic structures defined in [11].

In [8], planar graphs with extra information known as *marked graph diagrams* (also sometimes called *marked vertex diagrams* or *ch-diagrams*) were introduced as way of encoding knotted and linked surfaces

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in $\mathbb{R}^4$, known as *surface-links*. In [13] a set of moves on marked graph diagrams encoding ambient isotopy of surfaces in $\mathbb{R}^4$ analogous to the Reidemeister moves was proposed. In [7], generating sets of these *Yoshikawa moves* which are necessary and sufficient for ambient isotopy of oriented surface-links were identified.

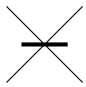
In this paper we define biquasile counting invariants for oriented surface-links and use them to define new nonnegative integer-valued invariants of oriented surface-links. We then enhance these invariants with *Boltzmann weights* taking values in a commutative ring R such that the multiset of Boltzmann weight values over the complete set of biquasile colorings of a marked graph diagram defines a stronger invariant of surface-links from which we can recover the biquasile counting invariant by taking the cardinality of the multiset. In particular, these invariants potentially provide obstructions to cobordism between classical knots and links. As this paper was nearing completion, the authors learned that similar results have been independently obtained and recently presented by Maciej Niebrzydowski [10].

The paper is organized as follows. In Section 2 we review the basics of marked graph diagrams and surface-links. In Section 3 we review biquasiles and define the biquasile counting invariant for surface-links. We provide examples to show that biquasile colorings can distinguish surface-links and can detect orientation reversals. In Section 4 we recall the biquasile Boltzmann weight enhancement and extend it to the case of oriented surface-links with some examples. As an application we show that these invariants can distinguish non-isotopic cobordisms between links. We end in Section 5 with some open questions for future research.

2. Marked graph diagrams

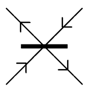
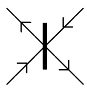
In this section, we review (oriented) marked graph diagrams representing surface-links.

Definition 1. A *marked graph* is a 4-valent graph in $\mathbb{R}^3$ each of whose vertices has a marker that looks like

 .

Two marked graphs are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhoods of markers. A marked graph in $\mathbb{R}^3$ can be described by a link diagram on $\mathbb{R}^2$ with some 4-valent vertices equipped with markers, called a *marked graph diagram*.

Definition 2. An *orientation* of a marked graph G in $\mathbb{R}^3$ is a choice of an orientation for each edge of G. An orientation of a marked graph G is said to be *consistent* if every vertex in G looks like

 or  .

A marked graph G in $\mathbb{R}^3$ is said to be *orientable* if G admits a consistent orientation. Otherwise, it is said to be *non-orientable*.

Definition 3. By an *oriented marked graph*, we mean an orientable marked graph in $\mathbb{R}^3$ with a fixed consistent orientation. Two oriented marked graphs are said to be *equivalent* if they are ambient isotopic in $\mathbb{R}^3$ with keeping the rectangular neighborhood of the marker and consistent orientation.

For $t \in \mathbb{R}$, we denote by $\mathbb{R}^3_t$ the hyperplane of $\mathbb{R}^4$ whose fourth coordinate is equal to $t \in \mathbb{R}$, i.e., $\mathbb{R}^3_t = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 = t\}$. A surface-link $\mathcal{L} \subset \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ can be described in terms of its *cross-sections* $\mathcal{L}_t = \mathcal{L} \cap \mathbb{R}^3_t,\ t \in \mathbb{R}$ (cf. [2]).