Topology and its Applications 235 (2018) 1–13

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Namioka spaces, GO-spaces and an o-game

Volodymyr Mykhaylyuk [a,b,*]

[a] *Jan Kochanowski University in Kielce, Poland*
[b] *Yurii Fedkovych Chernivtsi National University, Ukraine*

ARTICLE INFO

Article history:
Received 24 August 2017
Received in revised form 20 November 2017
Accepted 21 November 2017
Available online 2 December 2017

MSC:
primary 54C30, 54E52, 54F05
secondary 54C05, 54D30, 54D80

Keywords:
Namioka space
Separately continuous map
Linearly ordered space
GO-space
Topological game

ABSTRACT

We introduce and study a new modification of the classical weak Choquet game which we call by an o-game. We show that this game coincides with the σ-game in the class of GO-spaces. Moreover, we obtain a wide class non-Namioka Baire spaces and characterize Namioka spaces in terms of topological games in the class of GO-spaces which can be represented as union of ω_1 nowhere dense sets.

1. Introduction

It is well-known that a separately continuous function (i.e., function which is continuous with respect to each variable) $f : \mathbb{R}^2 \to \mathbb{R}$ can be jointly discontinuous. The question on a characterization of the discontinuity points sets of separately continuous functions $f : X \times Y \to \mathbb{R}$ defined on the product of topological spaces X and Y naturally arises. This investigation for case of $X = Y = [0, 1]$ was begun by R. Baire in [1] and was continued by many mathematicians for more general cases of spaces X and Y. The following Namioka's result has become a new impulse to development of these investigations.

Theorem 1.1 *(I. Namioka [12]). Let X be a strongly countably complete space, Y be a compact space and $f : X \times Y \to \mathbb{R}$ be a separately continuous function. Then there exists an everywhere dense G_δ-set A in X such that the function f is continuous at every point of the set $A \times Y$.*

* Correspondence to: Yurii Fedkovych Chernivtsi National University, Ukraine.
E-mail address: vmykhaylyuk@ukr.net.

https://doi.org/10.1016/j.topol.2017.11.008
0166-8641/

The following notion was introduced in [15].

A topological (Baire) space X is called *Namioka*, if for every compact space Y and every separately continuous function $f : X \times Y \to \mathbb{R}$ there exists a dense in X G_δ-set $A \subseteq X$ such that f is continuous at every point of $A \times Y$.

The topological game method is useful in many topics (see [3]). In particular, J.P.R. Christesen [4] used the topological game method for investigations of Namioka spaces for the first time.

Let $\mathcal{P}$ be a system of subsets of topological space X. We define a $G_{\mathcal{P}}$-game on X, in which the players α and β participate. A nonempty open in X set U_0 is the first move of β and a nonempty open in X set $V_1 \subseteq U_0$ and set $P_1 \in \mathcal{P}$ are the first move of α. Further, β chooses a nonempty open in X set $U_1 \subseteq V_1$ and α chooses a nonempty open in X set $V_2 \subseteq U_1$ and a set $P_2 \in \mathcal{P}$ and so on. The player α wins if $(\bigcap\limits_{n=1}^{\infty} V_n) \bigcap \overline{(\bigcup\limits_{n=1}^{\infty} P_n)} \neq \emptyset$. Otherwise, β wins.

A topological space X is called *α-favorable (β-favorable) in a topological game* if α (β) has a winning strategy in this game. A topological space X is called *α-unfavorable (β-unfavorable) in a topological-game* if α (β) has no winning strategy in this game. Clearly, any α-favorable topological space X is β-unfavorable.

In the case of $\mathcal{P} = \{X\}$ the game $G_{\mathcal{P}}$ is the classical Choquet game and X is β-unfavorable in this game if and only if X is a Baire space (see [13], [7], [15]). If $\mathcal{P}$ is the system of all finite (or one-point) subsets of X then $G_{\mathcal{P}}$-game is called a *σ-game*.

J. Saint-Raymond showed that every β-σ-unfavorable space is Namioka. A further development of this technique leads to a consideration of other topological games based on wider systems $\mathcal{P}$ of subsets of a topological space X.

Let T be a topological space and $\mathcal{K}(T)$ be the collection of all compact subsets of T. Then T is said to be *$\mathcal{K}$-countably-determined* if there exist a subset S of the topological space $\mathbb{N}^{\mathbb{N}}$ and a mapping $\varphi : S \to \mathcal{K}(T)$ such that for every open in T set $U \subseteq T$ the set $\{s \in S : \varphi(s) \subseteq U\}$ is open in S and $T = \bigcup\limits_{s\in S} \varphi(s)$; and it is called *$\mathcal{K}$-analytical* if there exists such a mapping φ for the set $S = \mathbb{N}^{\mathbb{N}}$.

A set A in a topological space X is called *bounded* if for any continuous function $f : X \to \mathbb{R}$ the set $f(A) = \{f(a) : a \in A\}$ is bounded.

The following theorem gives further generalizations of Saint-Raymond's result.

Theorem 1.2. *Any β-unfavorable in $G_{\mathcal{P}}$-game topological space X is Namioka if:*

(i) $\mathcal{P}$ is the system of all compact subsets of X (M. Talagrand [16]);
(ii) $\mathcal{P}$ is the system of all $\mathcal{K}$-analytical subsets of X (G. Debs [5]);
(iii) $\mathcal{P}$ is the system of all bounded subsets of X (O. Maslyuchenko [10]);
(iv) $\mathcal{P}$ is a system of all $\mathcal{K}$-countable-determined subsets of X (V. Rybakov [14]).

A topological space X is called *β-v-unfavorable* if X is a β-$G_{\mathcal{P}}$-unfavorable space for a system $\mathcal{P}$ of subsets of X which satisfies the following conditions:

(v_1) $\mathcal{P}$ is closed with respect to finite unions;
(v_2) for every $E \in \mathcal{P}$ and every compact $Y \subseteq C_p(X)$ the compact $\varphi(Y)$ is a Valdivia compact, where $\varphi : C_p(X) \to C_p(E)$, $\varphi(y) = y|_E$.

Theorem 1.2 was generalized in [11]. It was obtained in [11] that every β-v-unfavorable space is a Namioka space. Note that all these results give some necessary conditions on Namioka spaces. Concerning sufficient conditions we know two results only. J. Saint-Raymond in [15] showed that every completely regular Namioka space is Baire. M. Talagrand in [16] proved that for uncountable set I the space X of all functions $x : I \to$