Topology and its Applications 235 (2018) 22–34

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# Fibrations and global injectivity of local homeomorphisms ☆

L.R.G. Dias *, J. Venato-Santos

*Faculdade de Matemática, Universidade Federal de Uberlândia, Av. João Naves de Ávila 2121, 1F120 - CEP: 38408-100 Uberlândia, Brazil*

ARTICLE INFO

*Article history:*
Received 17 June 2016
Received in revised form 12 November 2017
Accepted 2 December 2017
Available online 5 December 2017

*MSC:*
14D06
58K15
57R30

*Keywords:*
Fibrations
Minimax methods
Global injectivity
Bijectivity

ABSTRACT

Given $X$ a path connected space and $g\colon X \to \mathbb{R}$ a local fibration on its image, we prove that $g$ satisfies a kind of deformation and consequently we obtain the path connectedness of its level sets. Then we provide global injectivity and invertibility theorems for local homeomorphisms $f\colon X \to \mathbb{R}^n$. These generalize known analytical results such as those given by Balreira and by Silva and Teixeira.



## 1. Introduction

Results on global injectivity and invertibility of locally injective mappings have been the object of intense research in the last decades. These results are related with interesting problems in several branches of Mathematics. We refer the reader to van den Essen [4] and Parthasarathy [10] for general presentations and connections with other problems.

The classical result of Hadamard states that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a $C^1$ local diffeomorphism, then $f$ is a diffeomorphism if, and only if, $f$ is a proper mapping. Thus, several metric and analytical conditions were presented in the literature to guarantee the properness condition. For instance, we have the following condition presented by Plastock [11]:

☆ LRGD acknowledges support from the Fapemig-Proc APQ-00431-14 grant. JV-S acknowledges support from the Fapemig-Proc APQ-00595-14 and the CNPq-Proc 446956/2014-7.

* Corresponding author.
*E-mail addresses:* lrgdias@ufu.br (L.R.G. Dias), jvenatos@ufu.br (J. Venato-Santos).

https://doi.org/10.1016/j.topol.2017.12.001
0166-8641/

$$\int_0^\infty \inf_{\|x\|=r} \|Df(x)^{-1}\|^{-1}\, dr = \infty.$$

For dimension two, the above condition was weakened by Sabatini [14] with the idea to control the topology of the level sets of the coordinate functions. He proved that a $C^1$ local diffeomorphism $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is globally injective if

$$\int_0^\infty \inf_{\|x\|=r} \|\nabla f_1(x)\|\, dr = \infty,$$

and $f$ is a bijective mapping if

$$\int_0^\infty \inf_{\|x\|=r} \frac{|\det Df(x)|}{\|\nabla f_1(x)\|}\, dr = \int_0^\infty \inf_{\|x\|=r} \frac{|\det Df(x)|}{\|\nabla f_2(x)\|}\, dr = \infty.$$

Recently, these last results were generalized to all dimensions by Balreira [1] (see Theorem 3.9 below).

Another approach to control the topology of the level sets of mappings is based on the use of regularity conditions at infinity, see for instance Rabier [13, §9] and Silva and Teixeira [15]. In this sense, Silva and Teixeira [15] introduced the notion of generalized Palais–Smale condition (cf. Definition 3.14) and they proved a result on global injectivity (see Theorem 3.15 below).

In our work, we show how the concept of *fibration* (cf. §2.1) can be applied to unify and to put forward the above discussion to a topological point of view. More precisely, we construct conditions, based on fibrations, which ensure global injectivity and invertibility of local homeomorphisms $f\colon X \to \mathbb{R}^n$, where $X$ is a path connected topological space.

In Subsection 2.1, we recall the definition and some properties of fibrations which are useful for our purposes and they are sufficient to prove the next result, which reveals the strengths of fibrations.

**Proposition 1.1.** *Let $X$ be a path connected set and let $f = (f_1, \dots, f_n)\colon X \to \mathbb{R}^n$ be a local homeomorphism. If $F_{n-1} = (f_1, \dots, f_{n-1}) : X \to \mathbb{R}^{n-1}$ is a local fibration, then $f$ is globally injective.*

In the above proposition, the condition that $F_{n-1}\colon X \to \mathbb{R}^{n-1}$ is a fibration cannot be weakened to fibration on its image (Definition 2.1), as can be viewed by Example 2.5. Thus, we present a generalization of the above proposition by means of fibrations on their images as follows:

**Theorem 1.2.** *Let $X$ be a path connected set and let $f = (f_1, \dots, f_n)\colon X \to \mathbb{R}^n$ be a local homeomorphism. If $F_1 = f_1$, $F_2 = (f_1, f_2)$, …, $F_{n-1} = (f_1, \dots, f_{n-1})$ are fibrations on their images, then $f$ is globally injective.*

This provides a direct extension of the bi-dimensional result by de Marco [9, Proposition 1] for any finite dimension. In the Subsection 2.2, we introduce the notion of *deformations on level sets of a function* (cf. Definition 2.6) which is motivated by Silva and Teixeira [15, Proposition 2.3]. Based on this definition, we establish a technical result on global injectivity (see Theorem 2.10), that in turn is the key to the proofs of our main results:

**Theorem 1.3.** *Let $X$ be a path connected space and let $f = (f_1, \dots, f_n)\colon X \to \mathbb{R}^n$ be a local homeomorphism. If $f_1$ and ${f_{k+1}}_{|F_k^{-1}(c_1,\dots,c_k)}$ are fibrations on their images, for any $k = 1, \dots, n-2$ and $(c_1, \dots, c_k) \in \mathbb{R}^k$, then $f$ is globally injective.*