



Almost complex structures for symplectic pairs

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ABSTRACT

Let M be a $2n$ -dimensional smooth manifold with a linear symplectic pair (ω, τ) which is a pair of skew-symmetric 2-forms of constant ranks with complementary kernel foliations. We study properties of almost complex structures compatible with the symplectic pair on M . We show that the space of compatible almost complex structures is contractible. This generalizes a result on compatible almost complex structures for symplectic manifolds.

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1. Introduction

The theme of symplectic topology concerns the global properties of symplectic manifolds. Symplectic forms, defined on even-dimensional smooth manifolds, are closed and non-degenerate 2-forms. It is well known that there is no local invariant for symplectic forms. For instance, Darboux theorem shows that any symplectic forms on $2n$ -dimensional manifolds are locally diffeomorphic to the standard symplectic form $\sum_{i=1}^n dy_i \wedge dx_i$ on \mathbb{R}^{2n} . Moser stability theorem shows that a family of cohomological symplectic forms can be isotopic to a constant one (see [1,13]). One of important methods of studying symplectic manifolds is the pseudo-holomorphic curves techniques. Some remarkable invariants, such as Floer homology, Fukaya category and Gromov–Witten invariants, based on the study of moduli spaces of pseudo-holomorphic curves which are smooth maps u from a Riemann surface with a complex structure (Σ, j) to symplectic manifold with an almost complex structure (M, J) such that $du \circ j = J \circ du$. The crucial reason why symplectic manifolds may contain many pseudo-holomorphic curves, which was observed by Gromov ([6]), is that every symplectic manifold admits plenty of compatible almost complex structures but needless to be integrable. Recall that an almost complex structure on a symplectic manifold (M, ω) is an automorphism J of the

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tangent bundle TM such that $J^2 = -id$. Such an almost complex structure is said to be compatible with the symplectic form ω if ω is J -invariant and $\omega(\cdot, J\cdot)$ is a Riemannian metric on M .

On the other hand, the non-degeneracy of symplectic forms plays a key role in developing relevant techniques for symplectic geometry. It is natural to consider what may happen if closed 2-forms might not be non-degenerate and whether some vital properties for almost complex structures still remain. In general, only one degenerate form may be not interesting enough. From Donaldson's work for four-manifolds ([5]), we know that a pair or triple of 2-forms may bring non-trivial geometric structures. For instance, a complex-symplectic structure is given by a pair of closed two forms θ_1, θ_2 such that $\theta_1^2 = \theta_2^2$ and $\theta_1 \wedge \theta_2 = 0$, and a hyperkähler structure is given by three closed two-forms $\theta_1, \theta_2, \theta_3$ such that $\theta_1^2 = \theta_2^2 = \theta_3^2$ and $\theta_i \wedge \theta_j = 0$ for $i \neq j$.

As a generalization of closed and non-degenerate 2-forms on symplectic manifolds, symplectic pairs are geometric objects also on even dimensional smooth manifolds. This geometric structure arises from the study of products of harmonic forms and of cohomology of symplectomorphism groups, introduced by Kotschick et al. ([3,9,10]). Given a $2n$ -dimensional smooth manifold M , a pair of closed 2-forms (ω, τ) defined on M is called a symplectic pair, if they have constant and complementary ranks, and ω restricts as symplectic form to the leaves of the kernel foliation \mathcal{F} of τ , τ restricts as symplectic form to the leaves of the kernel foliation \mathcal{G} of ω . Note that \mathcal{F} and \mathcal{G} are complementary smooth foliations ([14]). For instance, for a fixed integer d ($0 \leq d \leq n$), $(\mathbb{R}^{2n}, \omega_0, \tau_0)$ is the standard symplectic pair with rank $(2d, 2n - 2d)$, where $\omega_0 = \sum_{i=1}^d dy_i \wedge dx_i$, $\tau_0 = \sum_{j=d+1}^n dy_j \wedge dx_j$. Another obvious example is the product of two symplectic manifolds (M_1, η_1) and (M_2, η_2) which is a symplectic manifold $(M_1 \times M_2, \eta_1 \oplus \eta_2)$, at the same time, with natural structure of symplectic pair (ω, τ) , where $\omega = \eta_1 \oplus 0$, $\tau = 0 \oplus \eta_2$.

Parallelling to results for symplectic manifolds, some generalized versions of Darboux theorem, Moser stability theorem and neighbourhood theorems still hold for symplectic pairs ([2,7]). If we want to study much deeper invariants for symplectic pairs, especially using some kind of pseudo-holomorphic curves techniques, the almost complex structures compatible with symplectic pair are important objects.

In this article, as a first step to introduce almost complex geometry on symplectic pairs, we study the properties of compatible almost complex structure for symplectic pairs. Roughly speaking, given a symplectic pair (M, ω, τ) , an almost complex structure J on M is said to be *compatible* with symplectic pair (ω, τ) , if at every point $x \in M$, both the tangent spaces $Tf|_x$ and $Tg|_x$ of leaves of kernel foliations \mathcal{F} and \mathcal{G} passing x are J -invariant, and J is partially compatible with 2-forms ω and τ , respectively (see Definition 5.1).

Our main result is the following Theorem 1 which can be considered as a generalization of the result about compatible almost complex structures for symplectic forms to the case of symplectic pairs.

Theorem 1. *Let (M, ω, τ) be a $2n$ -dimensional smooth manifold with a symplectic pair (ω, τ) , $\mathcal{J}(M, \omega, \tau)$ the space of compatible almost complex structures on (M, ω, τ) . Then the space $\mathcal{J}(M, \omega, \tau)$ is contractible.*

To prove this theorem, we first study the properties of compatible complex structure for vector space with linear symplectic pairs, which can be considered as generalizations of the those about compatible complex structures for linear symplectic forms to the case of linear symplectic pairs. Next, we will consider symplectic pair vector bundles and verify the contractibility of associated space of compatible complex structures. Then the Theorem 1 (see Theorem 4) can be derived as a corollary.

2. Linear symplectic pairs

In this section, we first consider linear symplectic pairs, which are preliminary for the proof of our theorem.

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