Topology and its Applications 235 (2018) 73–82

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On functions that are almost continuous and perfectly everywhere surjective but not Jones. Lineability and additivity ☆

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ARTICLE INFO

Article history:
Received 16 November 2017
Received in revised form 2 December 2017
Accepted 4 December 2017
Available online 7 December 2017

MSC:
15A03
26A15
26A27
46J10

Keywords:
Lineability
Additivity
Perfectly everywhere surjective
Jones function
Almost continuous function
Bernstein set
Almost disjoint family
Covering Property Axiom CPA

ABSTRACT

We show that the class of functions that are perfectly everywhere surjective and almost continuous in the sense of Stallings but are not Jones functions is $\mathfrak{c}^+$-lineable. Moreover, it is consistent that this class is $2^\mathfrak{c}$-lineable, as this holds when $2^{<\mathfrak{c}} = \mathfrak{c}$. We also prove that the additivity number for this class is between ω_1 and $\mathfrak{c}$. This lower bound can be achieved even when $\omega_1 < \mathfrak{c}$, as it is implied by the Covering Property Axiom CPA. The main step in this proof is the following theorem, which is of independent interest: CPA implies that there exists a family $\mathcal{F} \subset C(\mathbb{R})$ of cardinality $\omega_1 < \mathfrak{c}$ such that for every $g \in C(\mathbb{R})$ the set $g \setminus \bigcup \mathcal{F}$ has cardinality less than $\mathfrak{c}$. Some open problems are posed as well.

☆ J.L. Gámez-Merino and J.B. Seoane-Sepúlveda were supported by grant MTM2015-65825-P.
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https://doi.org/10.1016/j.topol.2017.12.017
0166-8641/

1. Introduction

During a Math conference in Kent State University (Kent, OH) in November of 2016 the following question was posed to the public:

How "large" (in terms of algebraic genericity) is the class of functions in $\mathbb{R}^{\mathbb{R}}$ that are perfectly everywhere surjective and almost continuous (in the sense of Stallings) but not Jones?

More recently (in [19]) the study of the class of perfectly everywhere surjective functions that are not Jones was also considered (recall that Jones functions are, both, perfectly everywhere surjective and almost continuous).

The above question becomes clear once we define the following concepts.

Definition 1.1. Given a function $f\colon \mathbb{R} \to \mathbb{R}$, we say that:

(I) f is *perfectly everywhere surjective* ($f \in \mathrm{PES}$) if $f[P] = \mathbb{R}$ for every perfect set $P \subset \mathbb{R}$.
(II) f is a *Jones function* ($f \in \mathrm{J}$) if $C \cap f \neq \emptyset$ for every closed $C \subset \mathbb{R}^2$ with $\operatorname{dom} C$ (i.e., projection of C on the first coordinate) has cardinality continuum $\mathfrak{c}$.
(III) f is *almost continuous (in the sense of Stallings; $f \in \mathrm{AC}$)* if for each open set $G \subset \mathbb{R}^2$ such that $f \subset G$ there exists a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ such that $g \subset G$.

The notion of "being large" in terms of algebraic genericity is nowadays expressed in the following more precise terminology (see, e.g., [1,3,4,11–13,21,25,28]).

Definition 1.2. Given a (finite or infinite) cardinal number κ, a subset M of a vector space X is called κ-*lineable* in X if there exists a linear space $Y \subset M \cup \{0\}$ of dimension κ.

Intuitively, lineability seeks for a linear structure within $M \cup \{0\}$ of the highest possible dimension. However, there exist sets M containing no linear substructures of highest dimension, [4]. Due to the previous reason, this "maximal lineability number" is best expressed as the lineability coefficient $\mathcal{L}$ defined as the least cardinal for which there is *no* linear substructure of that cardinality (see [14] or [7]).

Definition 1.3. The *lineability coefficient* of a class $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ is defined as

$$\mathcal{L}(\mathcal{F}) = \min\{\kappa\colon \text{there is no } \kappa\text{-dimensional vector space } V \text{ with } V \subset \mathcal{F} \cup \{0\}\}.$$

Recall that $\mathcal{F}$ admits the maximal lineability number if, and only if, $\mathcal{L}(\mathcal{F})$ is a cardinal successor, that is, $\mathcal{L}(\mathcal{F})$ is of the form κ^+. (The symbol κ^+ stands for the successor cardinal of κ.) We refer the interested reader to [4,8–10,18,23] for many applications of this concept to several different fields within mathematics and, for a complete modern state of the art of this area of research, see [1,11].

On the other hand, and since the appearance of the work [22], the notion of lineability has been linked to that of the additivity coefficient A, which was introduced by the third author in [26,27].

Definition 1.4. Let $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$. The additivity of $\mathcal{F}$ is defined as the following cardinal number:

$$A(\mathcal{F}) = \min\bigl(\{\, |F| : F \subset \mathbb{R}^{\mathbb{R}} \wedge (\forall g \in \mathbb{R}^{\mathbb{R}})(g + F \not\subset \mathcal{F})\,\} \cup \{(2^{\mathfrak{c}})^+\}\bigr).$$

The class of all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ is denoted by $\mathrm{C}(\mathbb{R})$ Recall also that a function $f\colon \mathbb{R} \to \mathbb{R}$ is almost continuous if and only if it intersects every *blocking set*, that is, a closed set $K \subset \mathbb{R}^2$ which meets