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A special class of semi(quasi)topological groups and three-space properties [☆]

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ABSTRACT

The multiplication of a semitopological (quasitopological) group G is called sequentially continuous if the product map of $G \times G$ into G is sequentially continuous. In this paper, we mainly consider the properties of semitopological (quasitopological) groups with sequentially continuous multiplications and three-space problems in quasitopological groups. It is showed that (1) every snf-countable semitopological group G with the sequentially continuous multiplication is sof-countable; (2) if G is a sequential quasitopological group with the sequentially continuous multiplication, then G contains a closed copy of S_ω if and only if it contains a closed copy of S_2, which give a partial answer to a problem posed by R.-X. Shen; (3) let G be a quasitopological group with the sequentially continuous multiplication, then the following are equivalent: (i) G is a sequential α_4-space; (ii) G is Fréchet; (iii) G is strongly Fréchet; (4) (MA+¬CH) there exists a non-metrizable, separable, normal and Moore quasitopological group; (5) some examples are constructed to show that metrizability, first-countability and second-countability are not three-space properties in the class of quasitopological groups.

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1. Introduction

Recall that a *paratopological group* G is a group endowed with a topology such that the multiplication of G is jointly continuous. A *semitopological group* G is a group endowed with a topology such that the multiplication of G is separately continuous. A *topological group* (resp., *quasitopological group*) is a paratopological group (resp., semitopological group) G such that the inversion of G is continuous.

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As a generalization of topological groups, it is natural to consider which results valid for topological groups can be extended to semitopological groups or quasitopological groups [6]. Unfortunately, since the multiplication of a quasitopological group need not be continuous, some famous theorems of topological groups are not valid in quasitopological groups. For example, Comfort and Ross [11] proved that the product of an arbitrary family of pseudocompact topological groups is pseudocompact. However, C. Hernández and M. Tkachenko [21] constructed two pseudocompact quasitopological groups whose product fails to be pseudocompact. So it is suitable to consider a subclass of semitopological (quasitopological) groups.

The multiplication of a semitopological (quasitopological) group G is called *sequentially continuous* if the product map of $G \times G$ into G is sequentially continuous. It is equivalent to the condition that $a_n b_n \to e$ whenever $a_n \to e$ and $b_n \to e$, where e is the neutral element of the group G. Since the multiplication of a paratopological group G is continuous, it is sequentially continuous. Therefore, this subclass of semitopological (resp., quasitopological) groups contains paratopological (resp., topological) groups. In Section 3, we mainly consider semitopological groups with sequentially continuous multiplications. Some properties of this subclass of semitopological groups are obtained. We prove that every snf-countable semitopological group with the sequentially continuous multiplication is sof-countable. We also obtain some corollaries of this result.

In Section 4, the properties of quasitopological groups with sequentially continuous multiplications are discussed. It was proved in [31] that a topological group contains a (closed) copy of S_ω if and only if it contains a (closed) copy of S_2. R.-X. Shen pointed that there is a quasitopological group [35, Example 3.9] containing a closed copy of S_2. However, the quasitopological group contains no closed copy of S_ω. Therefore, the following problem was posed.

Problem 1.1. [35, Problem 3.11] Let G be a paratopological (quasitopological) group containing a closed copy of S_ω. Must G contain a closed copy of S_2?

We prove that if G is a sequential quasitopological group with the sequentially continuous multiplication, then G contains a closed copy of S_ω if and only if it contains a closed copy of S_2, which give a partial answer to Problem 1.1 for quasitopological groups. We also prove that every Fréchet quasitopological group with the sequentially continuous multiplication is strongly Fréchet. These results improve some relevant results in topological groups. At the end of this section, we construct under MA+¬CH a non-metrizable, separable, normal and Moore quasitopological group.

Let $\mathscr{P}$ be a (topological, algebraic, or mixed nature) property. We say that $\mathscr{P}$ is a *three-space property* if whenever a closed invariant subgroup N of a topological group G and the quotient group G/N have $\mathscr{P}$, so does G. Similarly one defines a three-space property in paratopological or quasitopological or semitopological groups. Three-space problems in topological groups are considered by many authors. The list of three-space properties in topological groups is quite long, it includes compactness, local compactness, pseudocompactness, precompactness, metrizability (first-countability), second-countability, connectedness, completeness, etc (see [6,9,10,12,27,38]). There are several papers which contain some results related to the three-space problem in the class of paratopological groups or semitopological groups (see [15,33,34,39]). Much less is known about three-space properties in quasitopological groups. Connectedness [39] and separability [15] are three-space properties in semitopological groups. It was pointed in [39] that compactness and local compactness are not three-space properties in quasitopological groups. M. Fernández and I. Sánchez showed that being a topological group is not a three-space property in the class of quasitopological groups [15, Example 2.9].

In Section 5, we continue the study of the three-space properties in the class of quasitopological groups. Some examples are constructed to show that metrizability, first-countability and second-countability are not three-space properties in the class of quasitopological groups.

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