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Soficity and hyperlinearity for metric groups

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ABSTRACT

We define sofic, weakly sofic, linear sofic and hyperlinear metric groups and discuss relationship among these classes.

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0. Introduction

Let us consider the class $\mathcal{G}$ of all complete metric groups (G, d) with bi-invariant metrics $d \leq 1$. Let $\mathcal{G}_{sof} \subset \mathcal{G}$ be the subclass consisting of all closed metric subgroups of metric ultraproducts of finite symmetric groups with normalized Hamming metrics. We call groups from $\mathcal{G}_{sof}$ *sofic metric groups*. We will prove in Theorem 1.1 below that there is an invariant metric on $\mathbb{Z}(p)$, $p \geq 13$, such that the corresponding metric group does not belong to $\mathcal{G}_{sof}$. In particular, $\mathcal{G}_{sof} \neq \mathcal{G}$. We emphasize that groups are considered together with metrics. Thus this inequality is a weaker version of the statement that there is a non-sofic countable group. The latter is a well-known conjecture in order to solve the Gromov's question of soficity of all countable groups (see [5], [6], [16]).

In fact our examples show that the class $\mathcal{G}_{sof}$ is a proper subclass of the class $\mathcal{G}_{w.sof} \subseteq \mathcal{G}$ consisting of closed metric subgroups of metric ultraproducts of finite metric groups with invariant metrics bounded by 1 [11]. We call it the class of *weakly sofic continuous metric groups*.

In a recent preprint [15] Nikolov, Schneider and Thom answering a question of Doucha from [8], show that $\mathcal{G}_{w.sof}$ is a proper subclass of $\mathcal{G}$. According to Section 3.2 of [7] this implies that the class $\mathcal{G} \setminus \mathcal{G}_{w.sof}$

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contains a finitely generated free group equipped with a bi-invariant discrete metric ≤ 1 (see Remark 2.7 below).

In our paper we define classes $\mathcal{G}_{hyplin} \subseteq \mathcal{G}$ and $\mathcal{G}_{l.sof} \subseteq \mathcal{G}$ of hyperlinear and linear sofic metric groups and discuss relationship among the classes of the collection

$$\{\mathcal{G}, \mathcal{G}_{sof}, \mathcal{G}_{w.sof}, \mathcal{G}_{hyplin}, \mathcal{G}_{l.sof}\}.$$

In view of the above we pay a special attention to discrete members of these classes. The results of Section 2 show that the corresponding classes of metric groups are closed under natural constructions of transformation of their members into discrete ones.

Although we mention below some connections of our results with continuous logic, our presentation of the material does not use continuous logic at all.

0.0.1. Preliminaries

In this paragraph we follow [10], [16] and [17]. Let G be a group. A function $l : G \to [0, \infty)$ is called a *pseudo length function* if

(i) $l(1) = 0$;
(ii) $l(g) = l(g^{-1})$;
(iii) $l(gh) \leq l(g) + l(h)$.

A *length function* is a pseudo length function satisfying

(i′) $l(g) = 0$ if and only if $g = 1$, where $g \in G$.

A pseudo length function is *invariant* if $l(h^{-1}gh) = l(g)$ for all $g, h \in G$. In this case it defines an invariant pseudometric d by $l(gh^{-1})$. It becomes a metric if l is a length function. Then we say that (G, l) is a *normed group*. It is an easy exercise that this notion is equivalent to the notion of *metric groups* (i.e. pairs (G, d)) with an invariant metric. It is worth noting that any unbounded bi-invariant norm l can be replaced by the norm $h \to \frac{l(h)}{1+l(h)}$ which defines the same topology with l. In this paper we consider normed groups with bounded norms.

Below we always assume that our metric groups are continuous structures with respect to bi-invariant metrics (see [4]). This exactly means that (G, d) is a complete metric space and d is bi-invariant. Note that the continuous logic approach takes weaker assumptions on d. Along with completeness it is only assumed that the operations of a structure are uniformly continuous with respect to d. Thus it is worth noting here that any group which is a continuous structure has an equivalent bi-invariant metric. See [13] for a discussion concerning this observation.

Metric ultraproducts of finite normed groups are deserved a particular attention in group theory. This is mainly motivated by investigations of *sofic groups*. We remind the reader that a group G is called *sofic* if G embeds into a metric ultraproduct of finite symmetric groups with the normalized Hamming distance d_H, [16]:

$$d_H(g, h) = 1 - \frac{|Fix(g^{-1}h)|}{n} \text{ for } g, h \in S_n.$$

A group G is called *hyperlinear* if G embeds into a metric ultraproduct of finite-dimensional unitary groups $U(n)$ with the normalized Hilbert–Schmidt metric d_{HS} (i.e. the standard l^2 distance between matrices), [9], [16]. It is an open question whether these classes are the same and whether every countable group is sofic/hyperlinear.

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