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A midpoint function and an end point function in continua

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ABSTRACT

In this paper we consider the hyperspace of arcs and singletons of a continuum and we study properties of an end point function and a midpoint function defined in such hyperspace. We investigate conditions under which such functions are continuous, open and/or closed, among other properties. Finally, we use the behavior of these functions to characterize some classes of continua.

1. Introduction

The study of hyperspaces has been of interest for a long time ([10], [11], [7]). Some of the most popular hyperspaces are: $CL(X)$ (the hyperspace of closed, nonempty subsets of X), 2^X (the hyperspace of compact, nonempty subsets of X), $C(X)$ (the hyperspace of subcontinua of X) and $F_n(X)$ (the hyperspace of nonempty subsets of X with at most n elements) for each $n \in \mathbb{N}$. In [11, p. 601] Nadler suggested to consider other hyperspaces; more precisely, he proposed to study the *hyperspace of arcs* of a continuum X, which is defined by $\mathcal{A}(X) = \{A \in C(X) : A \text{ is an arc in } X\}$ (by an *arc* we mean a space which is homeomorphic to the closed interval $[0, 1]$). In [13], Soto defined the *hyperspace of arcs and singletons* of a continuum X as the collection $\mathcal{M}(X) = \mathcal{A}(X) \cup F_1(X)$, and showed several properties of the hyperspace $\mathcal{M}(X)$ when the

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continuum X is a dendroid. This inspired Illanes to obtain a characterization of dendrites in terms of the hyperspace $\mathcal{M}(X)$ [6, Theorem 5]. In connection with this study, the authors of the present paper defined in [9] a natural function for $\mathcal{M}(X)$, which they called *midpoint function*. They also considered an end point function in $\mathcal{M}(X)$ and showed that the continuity of the midpoint function is equivalent to that of the end point function in continua [9, Theorem 4.12].

In this paper we study further properties of the midpoint and the end point functions and we use them to characterize some classes of continua. In Section 4 we analyze the continuity of the end point and the midpoint functions: we start giving some general conditions to obtain the continuity of such functions, as well as several examples that illustrate their behavior. Moreover, it is known that arcs and simple closed curves have continuous end point function ([6, p. 308] and [9, Theorem 3.16], respectively). In Section 4 we extend those results to regular continua and prove that further extensions may be hard to find.

In Section 5 we apply the results obtained in Section 4, and we present some characterizations of dendrites in terms of the behavior of the midpoint and the end point functions.

In Section 6 we investigate conditions under which the end point and/or midpoint functions belong to a particular class (e.g. one-to-one, homeomorphism, atomic, closed, open, monotone, confluent, weakly confluent, atriodic and light). As a by-product we characterize some classes of continua.

We conclude the paper with a short section of open questions.

2. Preliminaries

Throughout this paper, a *space* will mean a topological T_2 space.

For a subset A of a space X we denote the cardinality, the interior, the closure and the boundary of A in X by $|A|$, $int_X(A)$, $cl_X(A)$ and $bd_X(A)$, respectively. We say that a space is *nondegenerate* if it has more than one point.

A space X is *connected im kleinen* at a point $x \in X$ if each neighborhood of x contains a connected neighborhood of x. If a space is connected im kleinen at all of its points we say that it is *connected im kleinen* (see [12, 5.10]). It is well known that a space is connected im kleinen if and only if it is locally connected ([12, 5.22 (b)]).

A space X is *uniquely arcwise connected* if for each pair of distinct points $p, q \in X$ there exists a unique arc in X with end points p and q.

A *continuum* is a nonempty compact, connected, metric space. As usual, a *simple closed curve* is a continuum homeomorphic to the 1-sphere S^1 in the plane.

A continuum X is *unicoherent* if each pair of subcontinua of X whose union is X, has connected intersection. A continuum is *hereditarily unicoherent* if all its subcontinua are unicoherent.

A continuum is a *dendroid* if it is arcwise connected and hereditarily unicoherent. If X is a dendroid and $p \in X$ we will say that p is an *end point* of X if p is an end point of each arc in X that contains p. Moreover, p is a *ramification* point of X if it is the common end point of at least three otherwise disjoint arcs in X.

A *dendrite* is a locally connected continuum that contains no simple closed curve.

A continuum X is *regular* at a point $p \in X$ if each neighborhood of p in X contains an open neighborhood of p whose boundary in X is finite. When the continuum X is regular at all of its points, we say that X is a *regular* continuum.

A continuum is *arc free* if it contains no arcs.

Our next result, though simple, will be useful in the rest of the paper.

Lemma 2.1. *Let A be an arc in a continuum X. If U is an open subset of X that intersects A and $bd_X(U) \cap A$ has at most one point, then U intersects the set of end points of A.*