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Topologically independent sets in precompact groups

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ABSTRACT

It is a simple fact that a subgroup generated by a subset A of an abelian group is the direct sum of the cyclic groups $\langle a \rangle$, $a \in A$ if and only if the set A is independent. In [2] the concept of an *independent* set in an abelian group was extended to a *topologically independent set* in a topological abelian group (these two notions coincide in discrete abelian groups). It was proved that a topological subgroup generated by a subset A of an abelian topological group is the Tychonoff direct sum of the cyclic topological groups $\langle a \rangle$, $a \in A$ if and only if the set A is topologically independent and absolutely Cauchy summable. Further, it was shown, that the assumption of absolute Cauchy summability of A can not be removed in general in this result. In our paper we show that it can be removed in precompact groups.
In other words, we prove that if A is a subset of a *precompact* abelian group, then the topological subgroup generated by A is the Tychonoff direct sum of the topological cyclic subgroups $\langle a \rangle$, $a \in A$ if and only if A is topologically independent. We show that precompactness can not be replaced by local compactness in this result.

All groups in this paper are assumed to be abelian and all topological groups are assumed to be Hausdorff. A topological group is *precompact* if it is a topological subgroup of a compact group. As usually, the symbols $\mathbb{N}$ and $\mathbb{Z}$ stay for the sets of natural numbers and integers respectively.

Given an abelian group G, by 0_G we denote the zero element of G, and the subscript is omitted when there is no danger of confusion. Given a subset A of G, the symbol $\langle A \rangle$ stays for the subgroup of G generated by A. For $a \in G$, we use the symbol $\langle a \rangle$ to denote $\langle \{a\} \rangle$. Following [2], the symbol S_A stays for the direct sum

$$S_A = \bigoplus_{a \in A} \langle a \rangle,$$

and by $\mathcal{K}_A$ we denote the unique group homomorphism

$$\mathcal{K}_A : S_A \to G$$

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which extends each natural inclusion map $\langle a \rangle \to G$ for $a \in A$. As in [2], we call the map $\mathcal{K}_A$ the *Kalton map associated with* A.

We say that $\langle A \rangle$ *is a direct sum of cyclic groups* $\langle a \rangle$, $a \in A$ provided that the Kalton map $\mathcal{K}_A$ is an isomorphic embedding. When G is a topological group, we always consider $\langle a \rangle$ with the subgroup topology inherited from G and S_A with the subgroup topology inherited from the Tychonoff product $\prod_{a \in A} \langle a \rangle$. Finally, we say that $\langle A \rangle$ *is a Tychonoff direct sum of cyclic groups* $\langle a \rangle$, $a \in A$ if the Kalton map $\mathcal{K}_A$ is at the same time an isomorphic embedding and a homeomorphic embedding.

1. Introduction

The concept of compactness allows to transfer some purely non-topological issues into the realm of topology. A nice example of this phenomenon is the paper of Nagao and Shakhmatov (see [3]), where the classical, purely combinatorial result of Landau on the existence of kings in finite tournaments, where finite tournament means a finite directed complete graph, is generalized by means of continuous weak selections to continuous tournaments for which the set of players is a compact Hausdorff space. In our paper we provide another example of this phenomenon which non-trivially transfers a result from the area of abelian groups to the realm of precompact abelian groups.

Recall that a subset A of nonzero elements of a group G is *independent* provided that for every finite set $B \subset A$ and every family $(z_a)_{a \in B}$ of integers the equality $\sum_{a \in B} z_a a = 0$ implies $z_a a = 0$ for all $a \in B$.

Similarly, a subset A of nonzero elements of a topological group G is *topologically independent* (see [2, Definition 4.1]) provided that for every neighborhood W of 0_G there exists neighborhood U of 0_G such that for every finite set $B \subset A$ and every family $(z_a)_{a \in B}$ of integers the inclusion $\sum_{a \in B} z_a a \in U$ implies $z_a a \in W$ for all $a \in B$. This neighborhood U is called a *W-witness* of the topological independence of A.

One can readily verify that in (Hausdorff) topological groups every topologically independent set is independent (see [2, Lemma 4.2]) and that these two notions coincide in discrete groups. Thus topological independence can be viewed as a natural extension of independence.

Let us recall a basic and simple fact about independent sets.

Fact. A set A of nonzero elements of a group is independent if and only if the subgroup generated by A is a direct sum of cyclic groups $\langle a \rangle$, $a \in A$.

The aim of this paper is to prove the following counterpart of the above fact. Its proof is postponed to the end of the next section.

Theorem 1.1. *A set A of nonzero elements of a precompact group is topologically independent if and only if the topological subgroup generated by A is a Tychonoff direct sum of cyclic topological groups $\langle a \rangle$, $a \in A$.*

Example 3.1 demonstrates that precompactness can not be replaced by local compactness in Theorem 1.1.

In [2] a result closely related to Theorem 1.1 was obtained. In order to state it, recall that by [2, Definition 3.1] a subset A of a topological group is *absolutely Cauchy summable* provided that for every neighborhood U of 0_G

$$\text{there exists finite } F \subset A \text{ such that } \langle A \setminus F \rangle \subset U. \tag{1}$$

Let us state the promised result (see [2, Theorem 5.1]).

Fact 1.2. A subset A of nonzero elements of a topological group G is at the same time topologically independent and absolutely Cauchy summable if and only if the topological subgroup generated by A is a Tychonoff direct sum of cyclic topological groups $\langle a \rangle$, $a \in A$.