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Covering dimension, Bolzano and Steinhaus properties

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It is shown that the Bolzano property characterizes the covering dimension. Some relationships between the Steinhaus chain property and a dimension are discussed. Finally a new dimension for arbitrary topological spaces is defined.

1. Introduction

The classical Eilenberg–Otto theorem [3] on partitions of n-dimensional cubes is extended to obtain an elegant characterization of covering dimension of normal spaces in terms of existing essential families.

In 1817 Bolzano formulated an intermediate value theorem. Using the method of dividing intervals, he demonstrated that if a function f, continuous on a closed interval $[a, b]$, changes signs at the end points, then the value of this function equals zero in at least one point inside interval. Generalization of this result for n-dimensional cubes was given by Poincaré in 1883 [10]. For an arbitrary topological space the Bolzano property was defined by Kulpa [6], who proved that it characterizes the covering dimension in normal, countably paracompact spaces. In the present paper we will show that the Bolzano property, and the covering dimension are equivalent in the class of normal spaces.

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The Steinhaus' chessboard theorem was formulated in the book entitled "Mathematical snapshots" [12] in the following way: *let some segments of the chessboard be mined. Assume that the king cannot go across the chessboard from the left edge to the right one without meeting a mined square. Then the rook can go from upper edge to the lower one moving exclusively on mined segments.* According to Surówka [13] several proofs of the Steinhaus theorem seem to be incomplete or use induction on the size of chessboard. Kulpa, Socha and Turzański, assuming that the chessboard is divided into arbitrary polygons, gave an algorithm allowing to find rook's or king's route [7]. Tkacz and Turzański presented an n-dimensional version of this theorem, and proved that it is equivalent to the Poincaré theorem [15]. In the last section of Tkacz's paper [14] the reader can find the topological version of the Steinhaus chessboard theorem, which characterizes the Bolzano property in locally connected spaces. In our paper we will present two versions of Steinhaus chain property, and we will show that it gives a characterization of covering dimension in compact spaces as well as in normal, locally connected, and locally compact spaces. Finally we will give the definition of dimension, which is based on Steinhaus chain property.

2. Covering dimension

Definition 1. A family $\{(A_i, B_i) : i \in \Gamma\}$ of pairs of disjoint closed subsets is called Γ-*essential* if for every family $\{L_i : i \in \Gamma\}$, where L_i is a partition between A_i and B_i for every $i \in \Gamma$, we have $\bigcap\{L_i : i \in \Gamma\} \neq \emptyset$.

(A closed set L_i is a partition between A_i and B_i, if $X \setminus L_i$ is the union of two disjoint open sets, one containing A_i and the other one containing B_i.)

If $\Gamma = \{1, \dots, n\}$ we simply say that X has an n-essential family.

Theorem 1. [4, p. 35 and p. 78] *Let X be a normal space, then $dimX \geq n$ if and only if there exists an n-essential family.*

3. Bolzano property

Definition 2. [6, p. 91] The topological space X has a Γ-*Bolzano–Kulpa property* (Γ-BK property) if there exists a family $\{(A_i, B_i) : i \in \Gamma\}$ of pairs of disjoint closed subsets such that if $f : X \to R^\Gamma$ is a continuous map satisfying

$$p_i(f(A_i)) \subset (-\infty, 0], \text{ and } p_i(f(B_i)) \subset [0, \infty), \text{ for each } i \in \Gamma,$$

then there exists $c \in X$ such that $f(c) = 0$. The family $\{(A_i, B_i) : i \in \Gamma\}$ is called a Γ-*Bolzano–Kulpa system* (Γ-BK system).

Definition 3. The topological space X has a Γ-*Bolzano property* (denoted Γ-B property) if there exists a family $\{(A_i, B_i) : i \in \Gamma\}$ of pairs of disjoint closed subsets such that if $\{(H_i^-, H_i^+) : i \in \Gamma\}$ is a family of closed sets such that for each $i \in \Gamma$

$$A_i \subset H_i^-, B_i \subset H_i^+, \text{ and } H_i^- \cup H_i^+ = X,$$

then $\bigcap\{H_i^- \cap H_i^+ : i \in \Gamma\} \neq \emptyset$. The family $\{(A_i, B_i) : i \in \Gamma\}$ is called a Γ-*Bolzano system* (Γ-B system).

If $\Gamma = \{1, \dots, n\}$ we simply say that X has an n-Bolzano–Kulpa or n-Bolzano property, respectively.