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The class of Fedorchuk compact spaces is anti-multiplicative ☆

A.V. Ivanov

Institute of Applied Mathematical Research of Karelian Research Centre, Russian Academy of Sciences, Russian Federation

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ABSTRACT

A Hausdorff compact space is called a Fedorchuk compactum (or F-compactum) if it admits a decomposition into a special well-ordered inverse system with fully closed neighboring projections. It is known that the product of fully closed mappings is not fully closed, as a rule. We prove the same property for the class of Fedorchuk compacta: the product of F-compacta of spectral height 3 is never an F-compactum of countable spectral height.

The class of Fedorchuk compact spaces (or F-compacta) was defined by the author in 1984 [1] with the aim of clarifying the limits of the application of the method of fully closed mappings, which was developed by V. V. Fedorchuk (see [2]) and showed exceptional efficiency in constructing counterexamples in the general topology.

Definition 1. A Hausdorff compact space[1] X is called an F-compactum if there exists a well-ordered continuous inverse system $S = \{X_\alpha, \pi^\alpha_\beta : \alpha, \beta \in \gamma\}$ (here γ is the ordinal number) giving in the limit X, in which X_0 is a point, all neighboring projections $\pi^{\alpha+1}_\alpha$ are fully closed, $\alpha + 1 \in \gamma$, and inverse images of points $(\pi^{\alpha+1}_\alpha)^{-1}(x)$ are metrizable for any $x \in X\alpha$.

An inverse system satisfying the above conditions will be referred to hereafter as the F-system. The spectral height $sh(X)$ of an F-compactum X is the smallest length γ of the F-system S whose limit is equal to X.

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E-mail address: alvlivanov@krc.karelia.ru.

[1] We consider only compact Hausdorff spaces, all mappings are continuous.

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In [3] it is proved that for any countable ordinal number α such that $\alpha \neq \beta+1$ where β is a limit number, there exists an F-compactum X_α for which $sh(X_\alpha) = \alpha$.

F-compacta, in fact, these are compact spaces that can be constructed by the method of V.V. Fedorchuk. Recently interest to the class of F-compacta in connection with studies on renorming of the Banach space $C(X)$ has appeared. Namely, S.P. Gulko and M.S. Shulikina showed [4] that for every F-compact space X the space $C(X)$ admits an equivalent locally uniformly convex norm that is lower semicontinuous with respect to topology of pointwise convergence.

The class of fully closed maps on which the definition of F-compacta is based is antimultiplicative in the following sense: the product of fully closed mappings is not fully closed, as a rule (see [2], example 1.12). The main result of the paper is Theorem 1, which asserts that the class of F-compacta has the analogous property: the product of Fedorchuk compact spaces of spectral height 3 is never an F-compactum of a countable spectral height. Thus, we have obtained a positive answer to Question 1 of [5], in which it was shown, in particular, that the square of the Aleksandrov space "two arrows" (the F-compactum of spectral height 3) is not an F-compactum of the countable spectral height.[2]

Let's give the necessary definitions.

A surjective mapping $f : X \to Y$ is said to be fully closed (see [2]) if for any point $y \in Y$ and any finite open covering $O_1, \dots, O_k$ of the set $f^{-1}(y)$ the set

$$\bigcup_{i=1}^{k} f^\sharp(O_i) \cup \{y\}$$

is a neighborhood of the point y (here $f^\sharp(A) = \{y : f^{-l}(y) \subset A\}$, $A \subset X$).

Let $S = \{X_\alpha, \pi^\alpha_\beta : \alpha, \beta \in \gamma\}$ be an F-system, and let A be a closed subset of $X = \lim S$. Then the system

$$S_A = \{\pi_\alpha(A), \pi^\alpha_\beta|_{\pi_\alpha(A)} : \alpha, \beta \in \gamma\}$$

is also an F-system, because the restriction of a fully closed mapping to a closed subset is fully closed ([2], Proposition 1.14) (here π_α are the limit projections of the system S, if γ is a nonlimit ordinal, then $\lim S$ coincides with $X_{\gamma-1}$, and the limit projections π_α are projections $\pi^{\gamma-1}_\alpha$). It is known that $\lim S_A = A$, therefore a closed subset of an F-compactum is an F-compactum, and it is obvious that $sh(A) \leq sh(X)$.

The following statement was proved in [2], II.1.6:

Proposition 1. *Let $f : X \to Y$ be a fully closed mapping and A, B are disjoint closed subsets of X. Then $|f(A) \cap f(B)| < \omega_0$.*

We also need the following criterion for metrizability of the preimage of a metrizable compactum under a fully closed mapping, given in [2], Proposition 3.1:

Proposition 2. *Let $f : X \to Y$ be a fully closed mapping with metrizable fibers $f^{-1}(y)$, $y \in Y$, where Y is a metrizable compactum. Then the compactum X is metrizable if and only if*

$$|\{y : |f^{-1}(y)| > 1\}| \leq \omega_0.$$

The following three propositions are proved in [5]. For completeness of the presentation we give also their proofs.

[2] A consequence of this fact is the impossibility of representing the Helly space as an inverse limit of a countable system of resolutions – an answer to the S. Watson question (see [6], the problem 3.2.14).