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Inverting operations in operads

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ABSTRACT

We construct a localization for operads with respect to one-ary operations based on the Dwyer-Kan hammock localization [2]. For an operad $\mathscr{O}$ and a sub-monoid of one-ary operations $\mathcal{W}$ we associate an operad $L\mathscr{O}$ and a canonical map $\mathscr{O} \to L\mathscr{O}$ which takes elements in $\mathcal{W}$ to homotopy invertible operations. Furthermore, we give a functor from the category of $\mathscr{O}$-algebras to the category of $L\mathscr{O}$-algebras satisfying an appropriate universal property.

1. Introduction

If $\mathscr{O}$ is an operad in simplicial sets with n-ary operations $\mathscr{O}(n)$ and X is an algebra over $\mathscr{O}$, each n-ary operation gives a map

$$X^n \to X.$$

Unless X is quite trivial, we do not expect these maps to be invertible for $n \neq 1$. In contrast, one might like to insist that (at least some of) the one-ary operations are invertible (up to homotopy). To facilitate the

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study of such algebras, we seek an operad that encodes such information. This leads us to search for a good definition of localization for an operad $\mathcal{O}$ with respect to a submonoid $\mathcal{W} \subset \mathcal{O}(1)$ of one-ary operations.

Localizations have been much studied in the literature, particularly in the context of model categories. An especially useful and well-studied construction of the localization of a category is the hammock localization of Dwyer and Kan [2]. We propose a variant of their construction where we consider not only hammocks of string type, that is a sequence of right and left pointing arrows, but also of tree type where left and right pointing arrows are assembled in a tree. This seems a necessary complication so that operad composition is well defined and associative after localization.

Indeed, the complication arises because hammock localization does not preserve the monoidal structure of a category. It is well-known that operads (with an action of the symmetric group) correspond to strict (symmetric) monoidal categories with object set $\mathbb{N}$ where Hom-sets between two objects a and b are monoidally generated from Hom-sets with source 1. Since the hammock localization of [2] does not preserve the monoidal structure, the outcome does not readily provide an operad.

The proposed tree hammock localization

$$L^{TH}_{\mathcal{W}}(\mathcal{O}) \quad \text{(or simply } L\mathcal{O}\text{)}$$

of $\mathcal{O}$ is functorial in the pair $(\mathcal{O}, \mathcal{W})$ and every operation in $\mathcal{W}$ is invertible up to homotopy in $L\mathcal{O}$; see Lemmas 5.1 and 5.2 and Proposition 6.1. Furthermore the derived tensor product defines a functor from $\mathcal{O}$-algebras to $L\mathcal{O}$-algebras satisfying a natural universal property with respect to $\mathcal{O}$-algebra maps to $L\mathcal{O}$-algebras; see Proposition 6.2.

Our study has been motivated by considerations in topological and conformal quantum field theory. From Atiyah and Segal's axiomatic point of view, this is the study of symmetric monoidal functors from suitably defined cobordism categories. A particularly well-studied theory is the field theory modeled by the 1+1 dimensional cobordism category where the objects are disjoint unions of circles and the morphisms are (oriented) surfaces with boundary. Often one is led to the question of how the theory behaves stably, and more or less equivalently, when the operation defined by the torus is invertible. More generally there has been much recent interest in invertible topological field theories in the context of the study of anomalies and topological phases. See, for example, [3] and the references therein.

Much of topological field theory is captured when restricting to the (maximal) symmetric monoidal sub-category of the cobordism category corresponding to an operad. Therefore, the study of the surface operad is essential to the study of 1+1 dimensional topological field theory and conformal field theory. It has many homotopy equivalent models; one, denoted by $\mathcal{M}$, is built as a subcategory of Segal's category of Riemann surfaces [6] and we will keep this operad in mind as an example. This was also the motivating example for our study [1] of operads with homological stability, compare [8]. Indeed our discussion there led us to consider localizations of the operad $\mathcal{M}$ in an attempt to answer an old question of Mike Hopkins.[2]

Content

In Section 2, we review the definition of an operad and then, in Section 3, we characterize them as a subcategory in the category of strict symmetric monoidal categories. Next, in Section 4, we recall the definition and some properties of the standard hammock localization for categories from [2]. Our main new construction is the tree hammock localization in Section 5 where we also study several important properties. In Section 6 we provide a functor from algebras over a given operad to algebras over an associated localized operad.

[2] Stringy Topology in Morelia, Morelia, Mexico, 2006.