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Computational tools for topological coHochschild homology

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A B S T R A C T

In recent work, Hess and Shipley [18] defined a theory of topological coHochschild homology (coTHH) for coalgebras. In this paper we develop computational tools to study this new theory. In particular, we prove a Hochschild–Kostant–Rosenberg type theorem in the cofree case for differential graded coalgebras. We also develop a coBökstedt spectral sequence to compute the homology of coTHH for coalgebra spectra. We use a coalgebra structure on this spectral sequence to produce several computations.

1. Introduction

The theory of Hochschild homology for algebras has a topological analogue, called topological Hochschild homology (THH). Topological Hochschild homology for ring spectra is defined by changing the ground ring in the ordinary Hochschild complex for rings from the integers to the sphere spectrum. For coalgebras, there is a theory dual to Hochschild homology called coHochschild homology. Variations of coHochschild homology for classical coalgebras, or corings, appear in [7, Section 30], and [11], for instance, and the coHochschild complex for differential graded coalgebras appears in [17]. In recent work [18], Hess and Shipley define a topological version of coHochschild homology (coTHH), which is dual to topological Hochschild homology.

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In this paper we develop computational tools for coTHH. We begin by giving a very general definition for coTHH in a model category endowed with a symmetric monoidal structure in terms of the homotopy limit of a cosimplicial object. This level of generality makes concrete calculations difficult; nonetheless, we give more accessible descriptions of coTHH in an assortment of algebraic contexts.

One of the starting points in understanding Hochschild homology is the Hochschild–Kostant–Rosenberg theorem, which, in its most basic form, identifies the Hochschild homology of free commutative algebras. See [20] for the classical result and [29] for an analogue in the category of spectra. The setup for coalgebras is not as straightforward as in the algebra case and we return to the general situation to analyze the ingredients necessary for defining an appropriate notion of cofree coalgebras. In Theorem 3.11 we conclude that the Hochschild–Kostant–Rosenberg theorem for cofree coalgebras in an arbitrary model category boils down to an analysis of the interplay between the notion of "cofree" and homotopy limits. We then prove our first computational result, a Hochschild–Kostant–Rosenberg theorem for cofree differential graded coalgebras. A similar result has been obtained by Farinati and Solotar in [13] and [14] in the ungraded setting.

Theorem 1.1. *Let X be a nonnegatively graded cochain complex over a field k. Let $S^c(X)$ be the cofree coaugmented cocommutative coassociative conilpotent coalgebra over k cogenerated by X. Then there is a quasi-isomorphism of differential graded k-modules*

$$\mathrm{coTHH}(S^c(X)) \simeq \Omega^{S^c(X)|k},$$

where the right hand side is an explicit differential graded k-module defined in Section 3.

The precise definition of $\Omega^{S^c(X)|k}$ depends on the characteristic of k. Let $U(S^c(X))$ denote the underlying differential graded k-modules of the cofree coalgebra $S^c(X)$. If $\mathrm{char}(k) \neq 2$, we can compute $\mathrm{coTHH}(S^c(X))$ as $U(S^c(X)) \otimes U(S^c(\Sigma^{-1}X))$, while in characteristic 2, we have $\mathrm{coTHH}(S^c(X)) \cong U(S^c(X)) \otimes \Lambda(\Sigma^{-1}X)$, where Λ denotes exterior powers. Definitions of all of the terms here appear in Sections 2 and 3 and this theorem is proved as Theorem 3.15.

We then develop calculational tools to study $\mathrm{coTHH}(C)$ for a coalgebra spectrum C in a symmetric monoidal model category of spectra; see Section 4. Recall that for topological Hochschild homology, the Bökstedt spectral sequence is an essential computational tool. For a field k and a ring spectrum R, the Bökstedt spectral sequence is of the form

$$E^2_{*,*} = \mathrm{HH}_*(H_*(R;k)) \Rightarrow H_*(\mathrm{THH}(R);k).$$

This spectral sequence arises from the skeletal filtration of the simplicial spectrum $\mathrm{THH}(R)_\bullet$. Analogously, for a coalgebra spectrum C, we consider the Bousfield–Kan spectral sequence arising from the cosimplicial spectrum $\mathrm{coTHH}^\bullet(C)$. We call this the coBökstedt spectral sequence. We identify the E_2-term in this spectral sequence and see that, as in the THH case, the E_2-term is given by a classical algebraic invariant:

Theorem 1.2. *Let k be a field. Let C be a coalgebra spectrum that is cofibrant as an underlying spectrum. The Bousfield–Kan spectral sequence for* $\mathrm{coTHH}(C)$ *gives a* coBökstedt spectral sequence *with E_2-page*

$$E_2^{s,t} = \mathrm{coHH}^k_{s,t}(H_*(C;k)),$$

that abuts to

$$H_{t-s}(\mathrm{coTHH}(C);k).$$