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Classification of transversal Lagrangian stars

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A B S T R A C T

A Lagrangian star is a system of three Lagrangian submanifolds of the symplectic space intersecting at a common point. In this work we classify transversal Lagrangian stars in the symplectic space in the analytic category under the action of symplectomorphisms by using the method of algebraic restrictions. We present a list of all transversal Lagrangian star.

1. Introduction

The problem of classification of germs of s Lagrangian submanifolds $L_1, \cdots, L_s$ intersecting at a common point p (defined in [10] as s-Lagrangian star at p) under the action of symplectomorphisms was introduced by Janeczko in [10]. In the case of three Lagrangian subspaces in a symplectic vector space (M, ω) under the action of symplectic transformations, the natural invariant is the Maslov index ([11]), that is, the signature of the Kashiwara quadratic form $Q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$ defined on the direct sum of the Lagrangian subspaces. Janeczko generalizes the Maslov index to the nonlinear case.

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The aim of this paper is to obtain the symplectic classification of 3-Lagrangian stars two by two transversal in a symplectic space. For this purpose we use the method of algebraic restrictions introduced in [6]. We obtain a list of all transversal Lagrangian star.

A generalization of the Darboux–Givental Theorem ([2]) to germs of quasi-homogeneous subsets of the symplectic space was obtained in [6] and reduces the problem of symplectic classification of germs of quasi-homogeneous subsets to the problem of classification of algebraic restrictions of symplectic forms to these subsets. By this method, complete symplectic classifications of the $A-D-E$ singularities of planar curves and the S_5 singularity were obtained in [6].

The method of algebraic restrictions was used to study the local symplectic algebra of 1-dimensional singular analytic varieties. It is proved in [3] that the vector space of algebraic restrictions of closed 2-forms to a germ of 1-dimensional singular analytic variety is a finite-dimensional vector space.

The method of algebraic restrictions was also applied to the zero-dimensional symplectic isolated complete intersection singularities (see [4]) and to other 1-dimensional isolated complete intersection singularities: the S_μ symplectic singularities for $\mu > 5$ in [8], the $T_7 - T_8$ symplectic singularities in [9], the $W_8 - W_9$ symplectic singularities in [13] and the U_7, U_8 and U_9 symplectic singularities in [14]. In [7] the method is used to construct a complete system of invariants in the problem of classifying singularities of immersed k-dimensional submanifolds of a symplectic $2n$-manifold at a generic double point. In [1], the authors studied the local symplectic algebra of curves with semigroup $(4, 5, 6, 7)$ by this method.

This paper is organized as follows. Section 2 contains basic definitions about Lagrangian stars and the formulation of the main result. We also explain why we use the method of algebraic restrictions for this problem. We recall the method of algebraic restrictions in Section 3. In Section 4 we reduce the problem of classification of algebraic restrictions of symplectic forms to the linear case. Finally in Section 5 we obtain the symplectic classification of 3-Lagrangian stars two by two transversal.

2. Lagrangian stars

Consider $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^{n} dx_i \wedge dy_i)$ the $2n$-dimensional symplectic space with coordinate system $(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Let $\{L_1, ..., L_s\}$ be a system of Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega)$ intersecting at the origin.

Definition 2.1. ([10]) The germ of Lagrangian submanifolds $(\{L_1, \ldots, L_s\}, 0)$ is called s**-Lagrangian star**. If $s = 2$ and L_1 is transversal to L_2 then the 2-Lagrangian star $(\{L_1, L_2\}, 0)$ is called the basic Lagrangian star. The 3-Lagrangian star is simply called a Lagrangian star. We denote $L = L_1 \cup \cdots \cup L_s$.

Definition 2.2. The germ of a subset $N \subset (\mathbb{R}^m, 0)$ is called quasi-homogeneous if there exist a local coordinate system $x_1, \ldots, x_m$ of $(\mathbb{R}^m, 0)$ and positive integers $\lambda_1, \ldots, \lambda_m$ with the following property: if $(a_1, \ldots, a_m) \in N$ then $(t^{\lambda_1} a_1, \ldots, t^{\lambda_m} a_m) \in N$, for all $t \in [0, 1]$. The integers $\lambda_1, \ldots, \lambda_m$ are called weights of the variables $x_1, \ldots, x_m$, respectively.

Let $E = (\{L_1, \ldots, L_s\}, 0)$ be an s-Lagrangian star. We call E a *quasi-homogeneous* s-Lagrangian star if $L = L_1 \cup \cdots \cup L_s$ is a germ of a quasi-homogeneous subset. Moreover, E is called transversal if $L_1, \ldots, L_s$ are two by two transversal intersecting only at the origin.

Given $E = (\{L_1, \ldots, L_s\}, 0)$ and $E' = (\{L'_1, \ldots, L'_s\}, 0)$ two s-Lagrangian stars we say that they are diffeomorphic if there exists a germ of diffeomorphism $\Phi : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^{2n}, 0)$ such that $\Phi(L_i) = L'_{j_i}$ for some permutation j_i of $\{1, \ldots, s\}$. When Φ is a germ of a symplectomorphism of $((\mathbb{R}^{2n}, \omega), 0)$ we say that E and E' are *symplectically equivalent* (or equivalent).

The germ of a Lagrangian submanifold of $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dx_i)$ is symplectically equivalent to $L_1 = \{(x, y) \in \mathbb{R}^{2n} | x_1 = \cdots = x_n = 0\}$. The germ L_2 at 0 of a Lagrangian submanifold of $(\mathbb{R}^{2n}, \sum_{i=1}^{n} dx_i \wedge dx_i)$ which is transversal to L_1 at 0 can be described in the following way