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Symmetric products and closed finite-to-one mappings ☆

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ABSTRACT

In this paper, we continue the study of the symmetric products of generalized metric spaces in [39]. We consider the topological properties $\mathscr{P}$ such that the n-fold symmetric product $\mathscr{F}_n(X)$ of a topological space X has the topological properties $\mathscr{P}$ if and only if the space X or the product X^n does for each or some $n \in \mathbb{N}$. Depending on the operations under closed subspaces, finite products and closed finite-to-one mappings, two general stability theorems are obtained on symmetric products. We can apply the methods to unify and simplify the proofs of some old results in the literature and obtain some new results on symmetric products, list or prove 68 topological properties which satisfy the general stability theorems, and give answers to Questions 3.6 and 3.35 in [39].

1. Introduction

Borsuk and Ulam [14] introduced the notion of a symmetric product of an arbitrary topological space. For a topological space X and each $n \in \mathbb{N}$ the n-fold symmetric product $\mathscr{F}_n(X)$ can be obtained as a quotient space of Cartesian product X^n. For the closed unit interval $\mathbb{I}$, Borsuk and Ulam proved that the n-fold

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symmetric product $\mathscr{F}_n(\mathbb{I})$ is homeomorphic to $\mathbb{I}^n$ for each $n \in \{1, 2, 3\}$, and $\mathscr{F}_n(\mathbb{I})$ is not homeomorphic with any subset of Euclidean space $\mathbb{R}^n$ for any $n \geq 4$, also that $\dim \mathscr{F}_n(\mathbb{I}) = n$ for each $n \in \mathbb{N}$. Bott [15] showed that $\mathscr{F}_3(\mathbb{S}^1)$ is homeomorphic to $\mathbb{S}^3$, where $\mathbb{S}^1$ and $\mathbb{S}^3$ are the unit circle and the three-sphere, respectively. Later, Ganea [33], Molski [76], Schori [85], Macías [65–68] et al. further investigated the symmetric products.

Recently, Good and Macías [39] studied the symmetric products of generalized metric spaces. They obtained some generalized metric properties $\mathscr{P}$ such that for a topological space X and each $n \in \mathbb{N}$, the space $\mathscr{F}_n(X)$ has the property $\mathscr{P}$ if and only if X does. It turns out that the behavior of the symmetric product topology mirrors the behavior of the usual product topology. Most interesting, their methods are constructive and do not relay on operations under products and closed mappings, which reveal the inner construction of the spaces X and $\mathscr{F}_n(X)$. They gave some examples of spaces X satisfying some properties, but $\mathscr{F}_2(X)$ does not. The following questions were posed.

Question 1.1. [39, Question 3.6] If X is a Lašnev space, then is $\mathscr{F}_n(X)$ a Lašnev space for some integer $n \geq 2$?

Question 1.2. [39, Question 3.35] Let X be a space and $n \in \mathbb{N}$. If $\mathscr{F}_n(X)$ is a Morita's P-space, then is X a Morita's P-space?

Metrizability and compactness are the heart and soul of general topology. Also for applications, these two concepts are the most important: metric notions are used almost everywhere in mathematical analysis, while compactness is used in many parts of analysis and also in mathematical logic. Besides metrizability and compactness, there are a few other concepts which are fundamental in general topology, for examples, generalized metric spaces and covering properties [23,42].

In this paper, we continue to consider the symmetric products of generalized metric spaces and covering properties. What is a *generalized metric space*? The term is meant for classes which are 'close' to metrizable spaces in some sense: they usually possess some of the useful properties of metric spaces, and some of the theory or techniques of metric spaces carries over to these wider classes. To be most useful, they should be 'stable' under certain topological operations, e.g., finite or countable products, closed subspaces, and perfect (i.e., closed, with compact point-inverses) mappings [42, p. 425]. A topological property is called a *covering property* if it can be characterized by every open cover of a space having a certain refinement, for examples, compactness, Lindelöfness, paracompactness, subparacompactness, etc. To be most useful, they should be 'stable' under certain topological operations, e.g., closed subspaces, and closed or perfect mappings, but they may not be finite or countable productive.

Although Good and Macías pointed out that [39, p. 94]: "Where ever possible we have proved our results directly rather than relying on preservation under products and closed maps", we still try to depend on the operations under closed subspaces, finite products and closed finite-to-one mappings, and apply the methods to unify and simplify the proofs of some old results in the literature and obtain some new results on symmetric products. In Section 3, we consider the topological properties $\mathscr{P}$ such that a topological space X has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does by a general stability theorem on the images of X^n for each $n \in \mathbb{N}$ (see Theorem 3.1), list or prove 43 properties which satisfy the general stability theorem, and give an affirmative answer to Question 1.2 in Theorem 3.10. In Section 4, we consider the topological properties $\mathscr{P}$ such that for a topological space X and each $n \in \mathbb{N}$, the product X^n has the property $\mathscr{P}$ if and only if $\mathscr{F}_n(X)$ does by another general stability theorem on the images of X^n and the inverse images of $\mathscr{F}_n(X)$ (see Theorem 4.1), list or prove 25 topological properties which satisfy the general stability theorem, and Question 1.1 is negatively answered in Example 4.15.

A lot of topological properties satisfy the general stability theorems, this paper lists just a part of them.