Topology and its Applications 234 (2018) 69–81

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Strongly fibered objects and spaces

L. Stramaccia

Dipartimento di Matematica e Informatica, Università di Perugia, via Vanvitelli, 06100 Perugia, Italy

A R T I C L E   I N F O

Article history:
Received 4 September 2017
Received in revised form 11 November 2017
Accepted 12 November 2017
Available online 22 November 2017

Dedicated to the memory of Sibe Mardešić and Armin Frei

MSC:
55U35
55P55
18D20
18E35

Keywords:
Inverse system
Groupoid enriched category
Pseudo-natural transformation
Homotopy limit
Shape category
Strong shape category

A B S T R A C T

Given a model category $\mathcal{C}$ enriched over groupoids, we study and characterize the class of strongly fibered objects $\mathcal{F}(\mathcal{K})$ in terms of homotopy limits, with respect to a suitable subcategory $\mathcal{K}$ of $\mathcal{C}$. We also show that the strong shape category for the pair $(\mathcal{C}, \mathcal{K})$ turns out to be isomorphic to the usual shape category for the pair $(\mathcal{C}, \mathcal{F}(\mathcal{K}))$. This applies, in particular, to topological shape and strong shape.

1. Introduction

In the following $\mathcal{C}$ will denote a model category satisfying condition N of Edwards–Hastings [9], which is also enriched over the category Gpd of groupoids and let $\mathsf{Pro}\,\mathcal{C}$ denote the Grothendieck's category of inverse systems in $\mathcal{C}$. It is by now accepted that $\mathsf{Pro}\,\mathcal{C}$ can be equipped with a related model structure, where the classes of weak equivalences and cofibrations are essentially defined levelwise, see in chronological order [9], [18], [14]. The so called Steenrod homotopy category $\mathsf{Ho}\,(\mathsf{Pro}\,\mathcal{C})$ has been intensively studied in the past years and, for instance, it forms a central part in the encyclopedic book [16] by S. Mardešić, under the name of homotopy coherent category of inverse systems, being $\mathcal{C} = \mathsf{TOP}$ the category of topological spaces. In such a case TOP is equipped with the Strøm model structure where the weak equivalences are

E-mail address: luciano.stramaccia@unipg.it.

https://doi.org/10.1016/j.topol.2017.11.003
0166-8641/

the homotopy equivalences and the cofibrations are the Hurewicz cofibrations. We refer to [17] and [16] for all what concerns classical shape and strong shape.

A topological space X is said to be strongly fibered if it is the limit of an inverse system $\mathbf{X}$ of ANR-spaces, which is fibrant in the homotopy theory of ProTOP. The consideration of such spaces goes back to [11]. More recently they have appeared worth of consideration because of the following result [12], see also [7], corrected in [8]: a map $f : X \to Y$ is a strong shape equivalence for the pair $(\mathsf{TOP}, \mathsf{ANR})$ if and only if it is a shape equivalence for the pair $(\mathsf{TOP}, \mathcal{F}(\mathsf{ANR}))$. Here $\mathsf{ANR} \subset \mathsf{TOP}$ denotes the full subcategory of spaces having the homotopy type of an ANR-space (= absolute neighborhood retracts for metric spaces) and $\mathcal{F}(\mathsf{ANR})$ is the resulting subcategory of strongly fibered spaces.

There is a considerable amount of complexity and difficulty when passing from the shape category $\mathsf{Sh}(\mathsf{TOP}, \mathsf{ANR})$ to the strong shape category $\mathsf{SSh}(\mathsf{TOP}, \mathsf{ANR})$, the former being very categorical in nature while the latter, due to its more geometrical flavor, only recently had a rather satisfactory categorical interpretation [17], [22], [1]. Concerning the problem of homotopy coherence in connection with Strong Shape Theory one should also consider the papers [19], [6] and [5]. In two recent papers [21], [22] this author has given an alternative, more categorical, description both of the category $\mathsf{Ho}\,(\mathsf{Pro}\,\mathcal{C})$ and of the strong shape category $\mathsf{SSh}(\mathcal{C}, \mathcal{K})$, for $\mathcal{C}$ a category enriched over groupoids and $\mathcal{K}$ a suitable subcategory of $\mathcal{C}$. Based on such papers we introduce the category $\mathcal{F}(\mathcal{K})$ of strongly fibered objects of $\mathcal{C}$, with respect to $\mathcal{K}$, and prove, among other things, that it is the closure of $\mathcal{K}$ under homotopy limits. Moreover, we prove that the strong shape category $\mathsf{SSh}(\mathcal{C}, \mathcal{K})$ is isomorphic to the shape category $\mathsf{Sh}(\mathcal{C}, \mathcal{F}(\mathcal{K}))$.

2. Background

2.1. Groupoids

A *groupoid* is a small category whose morphisms are all invertible. The category Gpd of groupoids and their morphisms (functors) is a 2-category whose 2-cells are the natural isomorphisms.

The *component* of an element $x \in G$ of a groupoid is the set

$$[x] = \{y \in G \mid Hom_G(x, y) \neq \emptyset\}.$$

The functor $\pi_0 : \mathsf{Gpd} \to \mathsf{Set}$ assigns to every groupoid G the set of its components $\pi_0(G)$ and to a morphism $f : G \to H$ the function

$$\pi_0(f) : \pi_0(G) \to \pi_0(H),\ \ \pi_0(f)[x] = [f(x)].$$

For every $x \in G$, f also induces a group homomorphism $\pi_1(f) : \pi_1(x) \to \pi_1(f(x))$, where $\pi_1(x) = Hom_G(x, x)$ and $\pi_1(f)(\alpha) = f(\alpha)$.

$f : G \to H$ is a (categorical) equivalence if and only if both $\pi_0(f)$ and $\pi_1(f)$ are bijective (Whitehead's Lemma) [10], [20]. f will be called a π_0-*equivalence* when only $\pi_0(f)$ is a bijection.

2.2. ge-categories

A *ge-category* $\mathcal{C}$ is a category enriched over Gpd. $\mathsf{Gpd}(X, Y)$ will denote the hom-groupoid whose set of objects is $Hom_{\mathcal{C}}(X, Y)$. In other words a ge-category is a 2-category whose 2-cells are all invertible [3]. We call *maps* the 1-cells of $\mathcal{C}$ and *homotopies* its 2-cells, writing

$$\alpha : f \ \xrightarrow{\sim}\ g : X \to Y$$

to mean that α is a homotopy connecting the maps $f, g : X \to Y$. Homotopies in $\mathcal{C}$ can be composed both vertically $\beta \cdot \alpha$ and horizontally $\gamma * \alpha$.