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$\frac{1}{n}$-Homogeneity of the 2-nd cones

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A space is said to be $\frac{1}{n}$-homogeneous provided there are exactly n orbits for the action of the group of homeomorphisms of the space onto itself. In this paper, we investigate $\frac{1}{n}$-homogeneity in suspensions and cones of locally compact, homogeneous and finite dimensional metric spaces, we prove that if X is a solenoid, then the hyperspace of all subcontinua of X, is $\frac{1}{3}$-homogeneous. Moreover, we determine conditions under which the 2-nd cone of a Hausdorff space is $\frac{1}{2}$-homogeneous. Finally, we include a list of open problems related to this topic.

1. Introduction

Let $\mathcal{H}(X)$ denote the group of homeomorphisms of a space X onto itself. An *orbit of* X is the action of $\mathcal{H}(X)$ at a point x of X, namely $\{h(x) : h \in \mathcal{H}(X)\}$. The symbol $\mathcal{O}_X(x)$ denotes the orbit of space X that contains x. Given a positive integer n, a space X is a said to be $\frac{1}{n}$-*homogeneous* provided that X has exactly n orbits, in which case we say that the *degree of homogeneity of* X, denoted by $\mathrm{d_H}(X)$, is n (this notation was introduced in [34]). Observe that the family of orbits of X forms a decomposition of X; moreover, it follows immediately that the orbits of a space are homogeneous.

The notion of $\frac{1}{2}$-homogeneity is of particular importance since it is a blatant geometric property of every n-cell. In the last few years, important advances have been made in relation to this topic, for example, results about $\frac{1}{2}$-homogeneous continua appear in [11], [23], [26], [27], [28] and [32]. Moreover, results about

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$\frac{1}{2}$-homogeneity in certain classes of continua such as cones and suspensions are presented in [16], [17], [25], [30], [31] and [34]. Also, results about $\frac{1}{2}$-homogeneity in hyperspaces has been studied in [10], [19], [24] and [29]. Some results of this paper can be considered as a contribution in continuum theory.

This paper is organized as follows: In Section 2, we recall basic definitions and introduce some notation. In Section 3 we present some basic results on cones and suspensions and we introduce some important sets that will be used throughout the paper. In section 4 we present several examples of $\frac{1}{n}$-homogeneous spaces, some of the important results of this section are:

1. If X is a solenoid, then the hyperspace of all subcontinua of X is $\frac{1}{3}$-homogeneous.

2. We determine the degree of homogeneity of the cone of a locally compact, homogeneous, finite dimensional metric space X without isolated points when X is not connected.

3. We determine the degree of homogeneity of the cone of a locally compact, homogeneous, finite dimensional metric space X without isolated points when X is not locally contractible.

4. We determine the degree of homogeneity of the suspension of a locally compact, homogeneous, finite dimensional metric space X without isolated points when X is not connected.

Finally, some of the important results of the section 5 are:

1. Let X be a discrete space. Then $\mathrm{d_H}(\mathrm{Cone}(\mathrm{Sus}(X))) = 2$ if and only if $|X| \leq 2$.

2. Let X be a locally compact, homogeneous, finite dimensional metric space. If X is not locally contractible, then $\mathrm{d_H}(\mathrm{Cone}(\mathrm{Sus}(X))) = 4$.

3. Let X be a homogeneous, compact, metric space. If X is not locally connected, then $\mathrm{d_H}(\mathrm{Cone}(\mathrm{Sus}(X \times Q))) = 3$.

We end this paper with some open problems.

2. Notation and terminology

In this section we present general notation, we recall the concept of cone and suspension of a nonempty space. We also define terminology that we will use frequently. For notation and terminology not given here or in Section 1, see [22].

Part I. General notation:

The symbol $\mathbb{N}$ denotes the set of positive integers; $\mathbb{R}$ denotes the set of real numbers; $A \times B$ denotes the Cartesian product of A and B; $\overline{A}$ denotes the closure of A; iM and ∂M denote the interior and boundary manifolds, respectively, of a manifold M. Throughout the paper, I denotes the closed interval $[0, 1]$ and J denotes the closed interval $[-1, 1]$.

Part II. Quotient spaces:

Recall that for a topological space X, the cone of X, $\mathrm{Cone}(X)$, is the quotient space that is obtained by identifying all the points $(x, 1)$ in $X \times I$ to a single point ([22, p. 41, 3.15]). The suspension of X, $\mathrm{Sus}(X)$, is the quotient space that is obtained by identifying all the points $(x, 1)$ in $X \times J$ to a single point, and all the points $(x, -1)$ to another point [22, p. 42, 3.16]. Moreover, we denote the vertex of $\mathrm{Cone}(X)$ by v_X and the vertices of $\mathrm{Sus}(X)$ by v_X^1 and v_X^{-1}.

We often assume without saying so that $X \times (-1, 1)$ is a subspace of $\mathrm{Sus}(X)$. With this in mind, we write points in $\mathrm{Sus}(X)$ that are not the vertices as ordered pairs (x, t). Also, we consider $\mathrm{Cone}(X)$ as a subspace of $\mathrm{Sus}(X)$. When $A \subset X$, we consider $\mathrm{Sus}(A)$ as a subspace of $\mathrm{Sus}(X)$ with the same vertices, v_X^1 and v_X^{-1}, as in $\mathrm{Sus}(X)$.

Part III. General terminology:

A *continuum* is a compact and connected metric space.

The term *nondegenerate* refers to a space that contains more than one point.

An *arc* is a space homeomorphic to the closed interval $[0, 1]$. An arc A in a space X is a *free arc in X* provided that $A \setminus \partial A$ is open in X.