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Modified defect relations of the Gauss map and the total curvature of a complete minimal surface

Pham Hoang Ha

Department of Mathematics, Hanoi National University of Education, 136, XuanThuy str., Hanoi, Viet Nam

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ABSTRACT

In this article, we propose some conditions on the modified defect relations of the Gauss map of a complete minimal surface M to show that M has finite total curvature.

1. Introduction

In 1988, Fujimoto ([3]) proved Nirenberg's conjecture that if M is a complete non-flat minimal surface in $\mathbb{R}^3$, then its Gauss map can omit at most 4 points, and there are a number of examples showing that the bound is sharp (see [18, p. 72–74]). He ([6]) also extended that result to the Gauss maps of complete minimal surfaces in $\mathbb{R}^m$ $(m > 3)$. For the case of a complete minimal surface with finite total curvature in $\mathbb{R}^3$, Osserman ([17]) proved its Gauss map can omit at most 3 points. We note that a complete minimal surface with finite total curvature to be called an algebraic minimal surface. Many results related to this topic were given (see [19], [12], [11], [13], [9] and [7] for examples). Moreover, Mo and Osserman ([15]) showed an interesting improvement of Fujimoto's result by proving that a complete minimal surface in $\mathbb{R}^3$ whose Gauss map assumes five values only a finite number of times has finite total curvature. After that, Mo ([14]) extended that result to the complete minimal surface in $\mathbb{R}^m$ $(m > 3)$. Recently, the author, Phuong

E-mail address: ha.ph@hnue.edu.vn.

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and Thoan ([8]) improved these results by giving some conditions on the ramifications of the Gauss map of a complete minimal surface M in $\mathbb{R}^m$ $(m \geq 3)$ to show that M has finite total curvature.

On the other hand, in 1983, Fujimoto ([2]) introduced the non-integrated defect relations for the Gauss map of a complete minimal surface which are similar to the defect relations given by R. Nevanlinna in his value distribution theory. After that, he also showed the modified defect relations for the Gauss maps of complete minimal surfaces to improve the previous results on the value distribution theory of the Gauss maps of complete minimal surfaces relating to the omitted-properties, ramification properties ([4], [5]). Recently, the author and Trao ([9]), the author ([7]) studied the non-integrated defect relations for the Gauss map of a complete minimal surface with finite total curvature in $\mathbb{R}^m$ $(m \geq 3)$. These are the strict improvements of all previous results of Fujimoto on the modified defect relations for the Gauss map of a complete minimal surface with finite total curvature in $\mathbb{R}^m$.

A natural question is whether we may show a relation between of the modified defect relations of the Gauss map and the total curvature of a complete minimal surface. In this article, we would like to give an affirmative answer for this question. More precisely, we introduce some conditions on the modified defect relations of the Gauss map of a complete minimal surface M to show that M has finite total curvature.

The article is organized as follows: In Section 2, we recall some notations on the modified defect for a nonconstant holomorphic map of an open Riemann into $\mathbb{P}^k(\mathbb{C})$. After that we introduce two main theorems of this article and using them give some previous known results on the value-distribution-theoretic properties for the Gauss maps of complete minimal surfaces (Theorem 2.8, 2.9, 2.10 and 2.11). In Section 3, we give some lemmas which are necessary to prove the main theorems. Specially, we prove the Lemma 3.10, which is a generalization of the main lemma of Fujimoto in [5] by replacing the general position condition of hyperplanes by the subgeneral position one. In the last of this section we also show a relation between the classical defect in value distribution theory of meromorphic functions and modified defect. We will complete the main theorems in Section 4 and 5. We present the proofs basing on the manners of the proofs of the main theorems in [14] and [8].

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2. Statements of the main results

Let M be an open Riemann surface and f a nonconstant holomorphic map of M into $\mathbb{P}^k(\mathbb{C})$. Assume that f has reduced representation $f = (f_0 : \cdots : f_k)$. Set $||f|| = (|f_0|^2 + \cdots + |f_k|^2)^{1/2}$ and, for each a hyperplane $H : \overline{a}_0 w_0 + \cdots + \overline{a}_k w_k = 0$ in $\mathbb{P}^k(\mathbb{C})$ with $|a_0|^2 + \cdots + |a_k|^2 = 1$, we define $f(H) := \overline{a}_0 f_0 + \cdots + \overline{a}_k f_k$.

Definition 2.1. We define the S-defect of H for f by

$$\delta^S_{f,M}(H) := 1 - \inf\{\eta \geq 0; \eta \text{ satisfies condition } (*)_S\}.$$

Here, condition $(*)_S$ means that there exists a $[-\infty, \infty)$-valued continuous subharmonic function u $(\not\equiv -\infty)$ on M satisfying the following conditions:

(C1) $e^u \leq ||f||^\eta$,
(C2) for each $\xi \in f^{-1}(H)$, there exists the limit

$$\lim_{z \to \xi} (u(z) - \min(\nu_{f(H)}(\xi), k) \log |z - \xi|) \in [-\infty, \infty),$$

where z is a holomorphic local coordinate around ξ and $\nu_{f(H)}$ is the divisor of $f(H)$.