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# Every $\Sigma_s$-product of $K$-analytic spaces has the Lindelöf $\Sigma$-property ☆

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Given compact spaces $X$ and $Y$, if $X$ is Eberlein compact and $C_{p,n}(X)$ is homeomorphic to $C_{p,n}(Y)$ for some natural $n$, then $Y$ is also Eberlein compact; this result answers a question posed by Tkachuk. Assuming existence of a Souslin line, we give an example of a Corson compact space with a Lindelöf subspace that fails to be Lindelöf $\Sigma$; this gives a consistent answer to another question of Tkachuk. We establish that every $\Sigma_s$-product of $K$-analytic spaces is Lindelöf $\Sigma$ and $C_p(X)$ is a Lindelöf $\Sigma$-space for every Lindelöf $\Sigma$-space $X$ contained in a $\Sigma_s$-product of real lines. We show that $C_p(X)$ is Lindelöf for each Lindelöf $\Sigma$-space $X$ contained in a $\Sigma$-product of real lines. We prove that $C_p(X)$ has the Collins–Roscoe property for every dyadic compact space $X$ and generalize a result of Tkachenko by showing, with a different method, that the inequality $w(X) \leq nw(X)^{Nag(X)}$ holds for regular spaces.



## 1. Introduction

It is a well-known fact (which follows from Th. IV.1.7 of [1]) that if $X$ is Eberlein compact, $Y$ is compact and $C_p(X)$ is homeomorphic to $C_p(Y)$, then $Y$ is also an Eberlein compact space. In this direction, we solve affirmatively a question posed in [21] by showing that if $X$ is Eberlein compact, $Y$ is compact and $C_{p,n}(X)$ is homeomorphic to $C_{p,n}(Y)$, for some $n \in \mathbb{N}$, then $Y$ is Eberlein compact. A question from [26] is solved consistently in the negative way by providing an example of a Lindelöf subspace of a Corson compact space which does not have the Lindelöf $\Sigma$-property.

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Following Tkachuk, we call a space $X$ *Gul'ko* if $C_p(X)$ has the Lindelöf $\Sigma$-property. The $\Sigma_s$-products introduced in [17] are relevant in the study of Gul'ko spaces, since Sokolov proved that a compact space $X$ is a Gul'ko compact space iff $X$ embeds into a $\Sigma_s$-product of real lines. Tkachuk proved in [24] that every $\Sigma_s$-product of compact spaces is Lindelöf $\Sigma$. In this context, we show that if every $\sigma$-product of a family of topological spaces is Lindelöf $\Sigma$, then every $\Sigma_s$-product of this family is Lindelöf $\Sigma$. By applying this result, we conclude that every $\Sigma_s$-product of $K$-analytic spaces is Lindelöf $\Sigma$. On the other hand, we extend the above result of Sokolov by showing that every Lindelöf $\Sigma$-space contained in a $\Sigma_s$-product of real lines is a Gul'ko space. In a parallel direction, we establish that $C_p(X)$ is Lindelöf for each Lindelöf $\Sigma$-space contained in a $\Sigma$-product of real lines.

Collins–Roscoe spaces where introduced in [3], together with other similar properties, to study conditions for the metrizability of topological spaces. Some of the results mentioned above in combination with some results from [23] imply that both spaces $\Sigma_s\mathbb{R}^T$ and $C_p(\Sigma_s\mathbb{R}^T)$ have the Collins–Roscoe property. The first result was already established in [24], but the second one is new. Also, we show that $C_p(X)$ has the Collins–Roscoe property for certain dense subspaces $X$ of products of cosmic spaces. As a consequence we show that $C_p(X)$ has the Collins–Roscoe property for every dyadic compact space $X$.

In [18], Tkachenko proved that if a regular Lindelöf $\Sigma$-space $Y$ is homeomorphic to a subspace of a separable Hausdorff space, then $w(Y) \leq \mathfrak{c}$. To prove this result Tkachenko established the inequality $w(X) \leq nw(X)^{Nag(X)}$ for every Tychonoff space $X$. The proof of this inequality heavily relies on the use of methods of $C_p$-theory. In the last part of this paper, we provide a direct proof of the relation $w(X) \leq nw(X)^{Nag(X)}$ for every regular space $X$.

## 2. Terminology and notation

If not specified otherwise, all spaces in this article are assumed to be Tychonoff (that is, completely regular and Hausdorff). The space $\mathbb{R}$ is the real line with the usual order topology and $I = [0, 1]$ with the topology of subspace; the set $\omega \setminus \{0\}$ is denoted by $\mathbb{N}$ and 2 is the doubleton $\{0, 1\}$ with the discrete topology.

Given a set $A$ in a space $X$, say that a family $\mathcal{N}$ of subsets of $X$ is an *external network* of $A$ in $X$ if for any $x \in A$ and any open subset $U$ of $X$ with $x \in U$ there exists $N \in \mathcal{N}$ such that $x \in N \subset U$. If $X$ is a space and $f : X \to Y$ is a continuous map, then a family $\mathcal{N}$ of subsets of $X$ is a *network for* $f$ if for any $x \in X$ and any open subset $U$ of $Y$ such that $f(x) \in U$, there exists $N \in \mathcal{N}$ such that $x \in N$ and $f(N) \subset U$. A family $\mathcal{N}$ of subsets of a space $X$ is a *network with respect to a cover* $\mathcal{C}$ of $X$, if for every $C \in \mathcal{C}$ and every open subset $U$ of $X$ with $C \subset U$ there exists $N \in \mathcal{N}$ such that $C \subset N \subset U$.

The *Nagami number* of a space $X$, denoted by $Nag(X)$, is the minimal cardinal $\kappa$ for which there exist a compact cover $K$ of the space $X$ and family $\mathcal{N}$ of subsets of $X$ of cardinality $\kappa$ which is a network for the cover $K$. If $\mathcal{N}$ is a network with respect to the cover $\mathcal{C} = \{\{x\} : x \in X\}$, then we say that $\mathcal{N}$ is a *network of* $X$. A space with a countable network is called *cosmic*. The *network weight of* $X$ is the cardinal invariant $nw(X) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network of } X\}$.

For any space $X$ we denote by $C_p(X)$ the set of all real-valued continuous functions on $X$ endowed with the topology of pointwise convergence.

Given a fixed point $a$ in $X = \prod_{t\in T} X_t$, for $x \in X$, define the *support* $supp(x)$ *of* $x$ as the set $\{t \in T : x(t) \neq a(t)\}$. The subspace $\Sigma(X, a) = \{x \in X : |supp(x)| \leq \omega\}$ of $X$ is called the $\Sigma$*-product* of the family $\{X_t\}_{t\in T}$ centered at the point $a$. Analogously, the subspace $\sigma(X, a) = \{x \in X : |supp(x)| < \omega\}$ of $X$ is called the $\sigma$*-product* of the family $\{X_t\}_{t\in T}$ centered at the point $a$. Lastly, given $n \in \omega$, we will consider the set $\sigma_n(X, a) = \{x \in X : |supp(x)| \leq n\}$.

A space $X$ is called *simple* if it has at most one non-isolated point.

A map $f : X \to Y$ is called *a condensation* if it is a continuous bijection; in this case we say that $X$ condenses onto $Y$. If $X$ condenses onto a subspace of $Y$, we say that $X$ *condenses into* $Y$.

All topological notions whose definitions are not stated explicitly here should be understood as in [5].