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Standard special generic maps of homotopy spheres into Euclidean spaces

Dominik J. Wrazidlo

Mathematisches Institut, Ruprecht-Karls-Universität Heidelberg, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany

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A B S T R A C T

A so-called special generic map is by definition a map between smooth manifolds all of whose singularities are definite fold points. It is in general an open problem posed by Saeki in 1993 to determine the set of integers p for which a given homotopy sphere admits a special generic map into $\mathbb{R}^p$.

By means of the technique of Stein factorization we introduce and study certain special generic maps of homotopy spheres into Euclidean spaces called *standard*. Modifying a construction due to Weiss, we show that standard special generic maps give naturally rise to a filtration of the group of homotopy spheres by subgroups that is strongly related to the Gromoll filtration. Finally, we apply our result to some concrete homotopy spheres, which in particular answers Saeki's problem for the Milnor 7-sphere.

1. Introduction

A smooth map f between smooth manifolds is traditionally called a *special generic map* if every singular point x of f is a definite fold point, i.e., f looks in suitable charts around x and $f(x)$ like the multiple suspension of a positive definite quadratic form (see Section 2.1).

For a closed smooth manifold M^n of dimension n, let $S(M^n)$ denote the set of all integers $p \in \{1, \ldots, n\}$ for which there exists a special generic map $M^n \to \mathbb{R}^p$. The following problem was posed by Saeki in [1, Problem 5.3, p. 177].

Problem 1.1. Study the set $S(M^n)$.

For orientable M^n, Eliashberg [2] showed that $n \in S(M^n)$ if and only if M^n is stably parallelizable. We are concerned with the case that $1 \in S(M^n)$, i.e., the case that M^n admits a special generic map into $\mathbb{R}$.

E-mail address: dwrazidlo@mathi.uni-heidelberg.de.

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Such a map is usually referred to as *special generic function*, and is nothing but a Morse function all of whose critical points are extrema. If M^n admits a special generic function, then every component of M^n is homeomorphic to S^n by a well-known theorem of Reeb [3], and is for $n \leq 6$ even known to be diffeomorphic to S^n. If Σ^n denotes an exotic sphere of dimension $n \geq 7$, then according to [1, (5.3.4), p. 177] (compare Remark 5.2) we have

$$\{1, 2, n\} \subset S(\Sigma^n) \subset \{1, 2, \ldots, n-4, n\}. \tag{1.1}$$

Special generic functions can also be used in the study of individual homotopy spheres. Indeed, a homotopy sphere is said to have *Morse perfection* $\geq p$ (see Section 2.3) if it admits a family of special generic functions smoothly parametrized by points of the unit p-sphere that is subject to an additional symmetry condition. In [4] it is shown that the notion of Morse perfection is related as follows to the celebrated *Gromoll filtration* (see Section 2.5) of a homotopy sphere.

Theorem 1.2. *If Σ^n is a homotopy sphere of dimension $n \geq 7$, then*

$$(\textit{Gromoll filtration of } \Sigma^n) - 1 \leq \textit{Morse perfection of } \Sigma^n.$$

As pointed out in [4], it is possible by means of algebraic K-theory to derive upper bounds for the Morse perfection of certain homotopy spheres in terms of the signature of parallelizable null-cobordisms. Consequently, Theorem 1.2 allows to draw conclusions on the Gromoll filtration of concrete homotopy spheres such as Milnor spheres (compare Proposition 5.3(ii)). From the point of view of differential geometry the upper bounds for the Morse perfection imply (see [4, p. 388]) that certain homotopy spheres do not admit a Riemannian metric with sectional curvature "pinched" in the interval $(1/4, 1]$. Note that this is precisely the pinching condition of the classical sphere theorem due to Rauch [5], Berger [6] and Klingenberg [7].

The present paper focuses on *standard* special generic maps (see Definition 3.3) by which we roughly mean special generic maps from homotopy spheres into Euclidean spaces that factorize nicely over the closed unit ball (the required technique of *Stein factorization* is recapitulated in Section 2.2). According to Corollary 3.9 a homotopy sphere that admits a standard special generic map into $\mathbb{R}^p$ will also admit such maps into $\mathbb{R}^1, \ldots, \mathbb{R}^{p-1}$. The *fold perfection* of a homotopy sphere Σ^n (see Definition 3.10) is defined to be the greatest integer p for which a standard special generic map $\Sigma^n \to \mathbb{R}^p$ exists.

In refinement of Theorem 1.2 our main result is the following

Theorem 1.3. *If Σ^n is a homotopy sphere of dimension $n \geq 7$, then*

(i) *Gromoll filtration of $\Sigma^n \leq$ fold perfection of Σ^n, and*
(ii) *(fold perfection of Σ^n) $- 1 \leq$ Morse perfection of Σ^n.*

Compared to Morse perfection the notion of fold perfection has the natural algebraic advantage that it gives rise to a filtration of the group of homotopy spheres by sub*groups* (see Remark 3.12).

The paper is organized as follows. In Section 2 we explain in detail all relevant notions and techniques. Section 3 introduces standard special generic maps and studies some of their properties. The proof of Theorem 1.3, which will be given in Section 4, is based on a modification of the original proof of Theorem 1.2. Finally, in Section 5, we discuss various applications of Theorem 1.3. Corollary 5.1 focuses on the impact to Problem 1.1, whereas Proposition 5.3 exploits known results about the Gromoll filtration and the Morse perfection of concrete exotic spheres such as Milnor spheres to extract information about their fold perfection. In particular, we obtain the answer to Problem 1.1 for the Milnor 7-sphere in Corollary 5.4.