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Non reduced plane curve singularities with $b_1(F) = 0$ and Bobadilla's question

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ABSTRACT

If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to x^r.

1. Introduction

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function germ. What can be said about functions whose Milnor fibre F has the property $b_i(F) = 0$ for all $i \geq 1$? If F is connected then f is non-singular and equivalent to a linear function by A'Campo's trace formula. The remaining question: *What happens if F is non-connected?* is only relevant for non-reduced plane curve singularities.

This question is related to a recent paper [3]. That paper contains a statement about the so-called Bobadilla conjectures [2] in case of plane curves. The invariant $\beta = 0$, used by Massey [4] should imply that the singular set of f is a smooth line.

In this note we give a short topological proof of a stronger statement.

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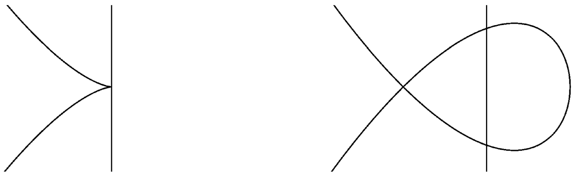

Fig. 1. Deformation to maximal number of double points.

Proposition 1.1. *If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to* x^r.

Corollary 1.2. *In the above case the singular set is a smooth line and the system of transversal singularities is trivial.*

2. Non-reduced plane curves

Non-isolated plane curve singularities have been thoroughly studied by Rob Schrauwen in his dissertation [5]. Main parts of it are published as [6] and [7]. The above Proposition 1.1 is an easy consequence of his work.

We can assume that $f = f_1^{m_1}. \cdots .f_r^{m_r}$ (partition in powers of reduced irreducible components).

Lemma 2.1. *Let* $d = \gcd(m_1, \cdots, m_r)$

(a.) *F has d components, each diffeomorphic to the Milnor fibre G of* $g = g_1^{\frac{m_1}{d}}. \cdots .g_r^{\frac{m_r}{d}}$. *The Milnor monodromy of f permutes these components,*
(b.) *if* $d = 1$ *then F is connected.*

Proof. (a.) Since $f = g^d$ the fibre F consists of d copies of G.

(b.) We recall here the reasoning from [5]. Deform the reduced factors f_i into $\hat{f}_i$ such that the product $\hat{f}_1. \cdots .\hat{f}_r = 0$ contains the maximal number of double points (cf. Fig. 1). This is called a network deformation by Schrauwen. The corresponding deformation $\hat{f}$ of f near such a point has local equation are of the form $x^p y^q = 0$ (point of type $D[p, q]$).

Near every branch $\hat{f}_i = 0$ the Milnor fibre is a m_i-sheeted covering of the zero-locus, except in the $D[p, q]$-points. We construct the Milnor fibre F of f starting with $S = \sum m_i$ copies of the affine line $\mathbb{A}$. Cover the ith branch with m_i copies of $\mathbb{A}$ and delete $(p + q)$ small discs around the $D[p, q]$-points. Glue in the holes $\gcd(p, q)$ small annuli (the Milnor fibres of $D[p, q]$). The resulting space is the Milnor fibre F of f.

A hyperplane section of a generic at a generic point of $\hat{f}_i = 0$ defines a transversal Milnor fibre $F_1^{\pitchfork}$. Start now the construction of F from $F_1^{\pitchfork}$, which consists of m_1 cyclic ordered points. As soon as $f_1 = 0$ intersects $f_k = 0$ it connects the sheets of $f_1 = 0$ modulo m_k. Since $\gcd(m_1, \cdots, m_r) = 1$ we connect all sheets. □

Proof of Proposition 1.1. If $b_1(F) = 0$, then also $b_1(G) = 0$. The Milnor monodromy has trace$(T_g) = 1$. According to A'Campo's observation [1] g is regular: $g = x$. It follows that $f = x^r$. □

3. Relation to Bobadilla's question

We consider first in any dimension $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with a 1-dimensional singular set, see especially the 1991-paper [8] for definitions, notations and statements.