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Singularities of tangent surfaces to directed curves

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A B S T R A C T

A directed curve is a possibly singular curve with well-defined tangent lines along the curve. Then the tangent surface to a directed curve is naturally defined as the ruled surface by tangent geodesics to the curve, whenever any affine connection is endowed with the ambient space. In this paper the local diffeomorphism classification is completed for generic directed curves. Then it turns out that the swallowtails and open swallowtails appear generically for the classification on singularities of tangent surfaces.

1. Introduction

Given a space curve, the ruled surface by its tangent lines is called a *tangent surface* or a *tangent developable* to the curve. Tangent surfaces appear in various geometric problems and applications (see for instance [2][8]). Even if the space curve is regular, its tangent surface has singularities at least along the original curve, so called "the curve of regression".

Let M be a general (semi-)Riemannian manifold, or more generally, a manifold M with an affine connection ∇, of dimension $m \geq 3$, and let $\gamma : I \to M$ any regular curve in M. If we replace tangent lines by "tangent geodesics" in the definition of tangent surface, then we have the definition of the ∇-tangent surface $\nabla\text{-Tan}(\gamma) : (I \times \mathbf{R}, I \times \{0\}) \to M$ as a map-germ along $I \times \{0\}$.

Ordinarily we try to classify certain generic singularities in a *specific* space, say, in the Euclidian spaces, in the space forms, and so on. If we treat arbitrary spaces, it would become hopeless to classify singularities of tangent surfaces that appear far away. However, it is possible to find a local classification theorem which

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holds in general spaces. In the previous paper [9], actually we have shown the following result on the singularities of ∇-tangent surfaces to generic curves for arbitrary affine connection ∇:

Theorem 1.1. ([9]) *The singularities of the ∇-tangent surface to a generic immersed curve in M on a neighborhood of the curve are only the cuspidal edges and the folded umbrellas if $m = 3$, and the embedded cuspidal edges if $m \geq 4$.*

The above theorem provides a rare but an ultimate *local* classification of singularities associated with *generic* immersed curves in *general* spaces. The explanation on singularities is coming later soon.

Now regarding the definition of general tangent surfaces, it seems to be very natural to consider the genericity in the space of curves, not only for regular (immersed) curves, but also for all singular curves with well-defined tangent directions, called *directed curves*, and to classify singularities of tangent surfaces for curves which are generic in such a class. In fact, as we show in this paper, it is possible and we have the following general result:

Theorem 1.2. (Singularities of tangent surfaces to generic directed curves.) *Let ∇ be any affine connection on a manifold M of dimension $m \geq 3$. The singularities of the ∇-tangent surface to a generic directed curve in M on a neighborhood of the curve are only the cuspidal edges, the folded umbrellas and the swallowtails if $m = 3$, and the embedded cuspidal edges and open swallowtails if $m \geq 4$.*

The genericity is exactly given by using Whitney C^∞ topology on an appropriate space of curves (see Proposition 4.1).

A map-germ $f : (\mathbf{R}^2, p) \to M$ is locally *diffeomorphic* at p to another map-germ $g : (\mathbf{R}^2, p') \to M'$ if there exist diffeomorphism-germs $\sigma : (\mathbf{R}^2, p) \to (\mathbf{R}^2, p')$ and $\tau : (M, f(p)) \to (M', g(p'))$ such that $\tau \circ f = g \circ \sigma : (\mathbf{R}^2, p) \to (M', g(p'))$.

The *cuspidal edge* is defined by the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^m, 0)$, $m \geq 3$,

$$(t, s) \mapsto (t + s,\ t^2 + 2st,\ t^3 + 3st^2,\ 0,\ \ldots,\ 0),$$

which is diffeomorphic to $(u, w) \mapsto (u, w^2, w^3, 0, \ldots, 0)$. The cuspidal edge singularities are originally defined only in the three dimensional space. Here we are generalizing the notion of the cuspidal edge in higher dimensional space. In Theorem 1.2, we emphasize it by writing "embedded" cuspidal edge. In what follows, we call it just cuspidal edge for simplicity even in the case $m \geq 4$. The *folded umbrella* (or the *cuspidal cross cap*) is defined by the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$,

$$(t, s) \mapsto (t + s,\ t^2 + 2st,\ t^4 + 4st^3),$$

which is diffeomorphic to $(u, t) \mapsto (u, t^2 + ut, t^4 + \frac{2}{3}ut^3)$. The *swallowtail* is defined by the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$

$$(t, s) \mapsto (t^2 + s,\ t^3 + \tfrac{3}{2}st,\ t^4 + 2st^2),$$

which is diffeomorphic to $(u, t) \mapsto (u, t^3 + ut, t^4 + \frac{2}{3}ut^2)$. The *open swallowtail* is defined by the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^m, 0)$, $m \geq 4$,

$$(t, s) \mapsto (t^2 + s,\ t^3 + \tfrac{3}{2}st,\ t^4 + 2st^2,\ t^5 + \tfrac{5}{2}st^3,\ 0,\ \ldots,\ 0),$$

which is diffeomorphic to $(u, t) \mapsto (u, t^3 + ut, t^4 + \frac{2}{3}ut^2, t^5 + \frac{5}{9}ut^3, 0, \ldots, 0)$. The open swallowtail singularity was introduced by Arnol'd (see [1]) as a singularity of Lagrangian varieties in symplectic geometry. Here we abstract its diffeomorphism class as the singularity of tangent surfaces (see [4][7]).