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Geometry of cuspidal edges with boundary

Luciana F. Martins [a], Kentaro Saji [b,*,1]

[a] *Departamento de Matemática, Instituto de Biociências, Letras e Ciências Exatas, UNESP – Univ Estadual Paulista, Câmpus de São José do Rio Preto, SP, Brazil*
[b] *Department of Mathematics, Graduate School of Science, Kobe University, Rokkodai 1-1, Nada, Kobe 657-8501, Japan*

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ABSTRACT

We study differential geometric properties of cuspidal edges with boundary. There are several differential geometric invariants which are related with the behavior of the boundary in addition to usual differential geometric invariants of cuspidal edges. We study the relation of these invariants with several other invariants.

1. Maps from manifolds with boundary

There are several studies for C^∞ map-germs $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ with $\mathcal{A}$-equivalence. Two map-germs $f, g : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphisms $\varphi : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ and $\Phi : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ such that $g \circ \varphi = \Phi \circ f$. There is also several studies for the case that the source space has a boundary. In [2], map-germs from 2-dimensional manifolds with boundaries into $\boldsymbol{R}^2$ are classified, and in [9], map-germs from 3-dimensional manifolds with boundaries into $\boldsymbol{R}^2$ are considered. Let $W \subset (\boldsymbol{R}^m, 0)$ be a closed submanifold-germ such that $0 \in \partial W$ and $\dim W = m$. We call $f|_W$ a *map-germ with boundary,* and we call interior points of W *interior domain of* $f|_W$. Since ∂W is an $(m-1)$-dimensional

* Corresponding author.
E-mail addresses: lmartins@ibilce.unesp.br (L.F. Martins), saji@math.kobe-u.ac.jp (K. Saji).
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submanifold, regarding $\partial W = B$, map-germs from manifolds with boundaries can be treated as a map-germ $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ with a codimension one oriented submanifold $B \subset (\boldsymbol{R}^m, 0)$. We consider that $(\boldsymbol{R}^m, 0)$ has an orientation and the submanifold B such that $0 \in B$. We define the interior domain of such a map-germ f as the component of $(\boldsymbol{R}^m, 0) \setminus B$ such that the positively oriented normal vectors of B point into it. With this terminology, an equivalent relation for map-germs with boundary is the following. Let $f, g : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ be map-germs with codimension one submanifolds $B, B' \subset (\boldsymbol{R}^n, 0)$ respectively, which contain 0. Then f and g are *$\mathcal{B}$-equivalent* if there exist an orientation preserving diffeomorphism $\varphi : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ such that $\varphi(B) = B'$, and a diffeomorphism $\Phi : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ that satisfies

$$g \circ \varphi = \Phi \circ f.$$

A map-germ $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ is a cuspidal edge if f is $\mathcal{A}$-equivalent to the map-germ $(u, v) \mapsto (u, v^2, v^3)$ at the origin. We say that f is a *cuspidal edge with boundary* $B \subset (\boldsymbol{R}^2, 0)$ if B is a codimension one oriented submanifold, that is, there exists a parametrization $b : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ to B satisfying $b'(0) \neq (0, 0)$. In this case, the domain which lies the left hand side of b with respect to the velocity direction is the interior domain of f.

In this note, we will consider differential geometric properties of cuspidal edges with boundaries. In order to do this, we first construct a normal form (Proposition 2.1) of it. It can be seen that all the coefficients of the normal form are differential geometric invariants. We give geometric meanings of these invariants. See [4,6,8,11,12,15–17,22], for example, for geometry of cuspidal edges itself. An application of this study is given by considering flat extensions of flat ruled surfaces with boundaries. See [13] for singularities of the flat ruled surfaces, and see [14] for flat extensions of flat ruled surfaces with boundaries. See [3] for flat extensions from general surfaces.

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2. Normal form of cuspidal edge with boundary

Now we look for a normal form of cuspidal edges with boundary. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a cuspidal edge with boundary $b : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$, $b'(0) \neq (0, 0)$. One can take a local coordinate system (u, v) on $(\boldsymbol{R}^2, 0)$ and an isometry Φ on $(\boldsymbol{R}^3, 0)$ satisfying that

$$\begin{aligned}&\Phi \circ f(u, v)\\ &= \Big(u,\ \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{1}{2}v^2,\ \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\Big) + h(u, v),\end{aligned} \tag{2.1}$$

where $b_{03} \neq 0$, $b_{20} \geq 0$, and

$$h(u, v) = \big(0,\ u^4 h_1(u),\ u^4 h_2(u) + u^2 v^2 h_3(u) + u v^3 h_4(u) + v^4 h_5(u, v)\big),$$

with $h_1(u)$, $h_2(u)$, $h_3(u)$, $h_4(u)$, $h_5(u, v)$ smooth functions. See [11] for details.

Now we consider $b(t) = (b_1(t), b_2(t))$. We have two cases.

(1) $b_1'(0) \neq 0$,
(2) $b_1'(0) = 0$, $b_2'(0) \neq 0$.

In the case (1), one can take u for the parameter of b and parametrized by

$$b(u) = \left(\varepsilon u,\ \sum_{k=1}^{3} \frac{c_k}{k!} u^k + u^4 c(u)\right) \quad (\varepsilon = \pm 1). \tag{2.2}$$