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Realization of graphs by fold Gauss maps

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We define here a special type of bipartite graph, called 2-negative, and prove that any 2-negative graph with total weight equal to zero can be associated with some fold Gauss map from a closed orientable surface.

1. Introduction

The Gauss map of a surface generically immersed in Euclidean 3-space is described in [2]. The singularities of a stable Gauss map, in Whitney's sense, being fold curves with isolated cusp points, are called the parabolic set of the surface. Each parabolic curve in this set separates a hyperbolic region from an elliptic region of the surface.

In order to study the global behavior of Gauss maps it is useful to codify all the information relative to the topological type of the complement of the parabolic set on the surface in the simplest possible way. In [10], the authors introduce the study of graphs with weights associated with stable Gauss maps, where it is

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shown that *any weighted bipartite graph can be associated to a stable Gauss map from an appropriate closed orientable surface.*

In the particular case of the parabolic set of a stable Gauss map having no cusp points, which is called a fold Gauss map, they also prove that the number of connected components of the parabolic curve (or equivalently the number of edges of the associated graph) is even. A natural question at this point is whether there is a special type of graph that can be associated to stable fold Gauss maps.

Our main objective here is to study the particular case of graphs with a total weight equal to zero. In Section 3, we introduce the definition of a 2-negative graph and in Section 5 we show that *a graph with total weight equal to zero is a graph corresponding to a fold Gauss map of a closed orientable surface if and only if it is a 2-negative graph.*

In order to prove this result we shall use an inductive constructive process starting from simple basic graphs of well known examples and then apply suitable codimension one transitions (see [3]) and surgeries (see [10]), that we describe in Section 4. The codimension one transitions of their singularities are determined through the study of the families of height functions associated to generic 1-parameter families of embeddings. The surgery of immersions on a closed surface consists in joining two elliptic regions by one hyperbolic region. By means of suitable combinations of surgeries and codimension one transitions, we can produce required immersions associated to Gauss maps. A useful factor in this process is the existence of basic graphs with total weight equal to zero (Fig. 8) and a process of suitable manipulation of the immersions of a closed orientable surface in Euclidean 3-space that generates the fold Gauss maps associated to these graphs. These basic graphs correspond to fold Gauss maps of a torus whose two parabolic curves separate a hyperbolic region from an elliptic disc.

2. Graphs of stable Gauss maps

Let M and N be smooth orientable surfaces and f, $g : M \longrightarrow N$ smooth maps between them. The maps f and g are $\mathcal{A}$-**equivalent** (or **equivalent**) if there are orientation-preserving diffeomorphisms, l and k, such that $g \circ l = k \circ f$. A map $f : M \longrightarrow N$ is said to be **stable** if all maps sufficiently close to f, in the Whitney C^∞-topology (see [5]), are equivalent to f.

A point of the source surface M is a **regular** point of f if the map f is a local diffeomorphism around that point and **singular** otherwise. We denote by Σf the **singular set** of a map f and its image $f(\Sigma f)$ is the **branch set** of f.

The concept of stability for a Gauss map of a surface immersed in $\mathbb{R}^3$ is slightly different from the general case of maps between surfaces in the sense that it must depend on perturbations of the immersion rather than on those of the map itself.

Since every surface decomposes into a disjoint union of connected surfaces, we will assume that the surfaces with which we work are connected.

Given an immersion $f : M \to \mathbb{R}^3$ of a closed orientable surface M in $\mathbb{R}^3$, let $\mathcal{N}_f : M \to S^2$ be its Gauss map. This map $\mathcal{N}_f$ is said to be **stable** if there exists a neighborhood $\mathcal{U}_f$ of f in the space $\mathcal{I}(M, \mathbb{R}^3)$ of immersions of M into $\mathbb{R}^3$ such that for all $g \in \mathcal{U}_f$, the Gauss map $\mathcal{N}_g$ associated to g is $\mathcal{A}$-equivalent to $\mathcal{N}_f$. It can be seen that this condition is equivalent to stating that the family of height functions associated to f:

$$\begin{array}{rcccl} \lambda(f) & : & M \times S^2 & \longrightarrow & \mathbb{R} \\ & & (x, v) & \longmapsto & \langle f(x), v\rangle = f_v(x) \end{array}$$

is structurally stable ([2], [11]).