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A note on distinguished bases of singularities ☆

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Dedicated to the memory of Egbert Brieskorn with great admiration

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For an isolated hypersurface singularity which is neither simple nor simple elliptic, it is shown that there exists a distinguished basis of vanishing cycles which contains two basis elements with an arbitrary intersection number. This implies that the set of Coxeter–Dynkin diagrams of such a singularity is infinite, whereas it is finite for the simple and simple elliptic singularities. For the simple elliptic singularities, it is shown that the set of distinguished bases of vanishing cycles is also infinite. We also show that some of the hyperbolic unimodal singularities have Coxeter–Dynkin diagrams like the exceptional unimodal singularities.

0. Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of an analytic function with an isolated singularity at the origin defining a singularity $(X_0, 0)$ where $X_0 = f^{-1}(0)$. We assume $n \equiv 3 \,(\mathrm{mod}\, 4)$. (This can be achieved by a stabilization of the singularity.) An important invariant of the singularity $(X_0, 0)$ is the symmetric bilinear intersection form $\langle \cdot, \cdot \rangle$ on the Milnor lattice of f. It is well known that this intersection form is negative definite if and only if the singularity is simple, it is negative semidefinite if and only if the singularity is simple elliptic and it is indefinite otherwise. It follows from the Cauchy–Schwarz inequality that for negative definite symmetric bilinear forms, the only values of $\langle x, y \rangle$ for non-collinear vectors x, y with $\langle x, x \rangle = \langle y, y \rangle = -2$ are

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$0, \pm 1$. For semidefinite symmetric bilinear forms, also the values ± 2 can be achieved. Another invariant of the singularity is the class of distinguished bases of vanishing cycles $\mathcal{B}^*$ of the singularity [6]. The intersection matrix with respect to a distinguished basis of vanishing cycles is encoded by a graph, a Coxeter–Dynkin diagram of the singularity. In this way, the class $\mathcal{B}^*$ gives rise to the class $\mathcal{D}^*$ of Coxeter–Dynkin diagrams with respect to distinguished bases of vanishing cycles. Here we consider questions related to the finiteness of these sets. N. A'Campo [1] has shown that if f has corank 2, then there exists a distinguished basis of vanishing cycles for f such that the mutual intersection numbers are only 0 or ± 1. Here we are interested in the opposite question: For which singularities do there exist distinguished bases of vanishing cycles such that there are pairs of basis elements with an arbitrary intersection number? We show that such bases exist for all isolated hypersurface singularities which are neither simple nor simple elliptic. This implies that for these singularities the sets $\mathcal{B}^*$ and $\mathcal{D}^*$ are infinite. For the simple singularities, both sets are finite. We show that for the simple elliptic singularities the set $\mathcal{D}^*$ is finite, but not the set $\mathcal{B}^*$. In this way, we obtain a characterisation of simple and simple elliptic singularities. For the proof, we show among other things that the hyperbolic singularities $T_{3,3,4}$, $T_{2,4,5}$, and $T_{2,3,7}$ have Coxeter–Dynkin diagrams like the exceptional unimodal singularities.

1. Distinguished bases of vanishing cycles

We briefly recall the definition of distinguished bases of vanishing cycles.

Let $f_\lambda : U \to \mathbb{C}$ be a *morsification* of f. This is a perturbation of (a representative of) f (i.e., $f_0 = f$) defined in a suitable neighbourhood U of the origin in $\mathbb{C}^n$, depending on a parameter $\lambda \in \mathbb{C}$ and such that, for $\lambda \neq 0$ small enough, the function f_λ has only non-degenerate critical points with distinct critical values. The number of these critical points is equal to the Milnor number μ of the germ f. Choose a small closed disc $\Delta \subset \mathbb{C}$ which contains all the μ critical values of f_λ in its interior. Let $\mathcal{X} := f_\lambda^{-1}(\Delta) \cap B_\varepsilon$ and $X_t := f_\lambda^{-1}(t) \cap B_\varepsilon$ for $t \in \Delta$ where B_ε is the ball of radius ε around the origin in $\mathbb{C}^n$. Assume that ε and $\lambda \neq 0$ are chosen so small that all the critical points of the function f_λ are contained in the interior of $\mathcal{X}$ and the fibre $f_\lambda^{-1}(t)$ for $t \in \Delta$ intersects the ball B_ε transversely. Choose a basepoint $s \in \mathbb{C}$ on the boundary of this disc. Join the critical values to the base point s by a system of non-self-intersecting paths $\gamma_1, \dots, \gamma_\mu$ in Δ meeting only at s and numbered in the order in which they arrive at s, where we count clockwise from the boundary of the disc (see, e.g., [10, Figure 5.3]). Each path γ_i gives a vanishing cycle $\delta_i \in H_{n-1}(X_s; \mathbb{Z})$ determined up to orientation. It satisfies $\langle \delta_i, \delta_i \rangle = -2$. After choosing orientations we obtain a system $(\delta_1, \dots, \delta_\mu)$ of vanishing cycles which is in fact a basis of $H_{n-1}(X_s; \mathbb{Z})$. Such a basis is called a *distinguished basis of vanishing cycles*. Let $\mathcal{B}^*$ be the set of distinguished bases of the singularity $(X_0, 0)$. There is an action of the braid group Z_μ in μ strings on the set $\mathcal{B}^*$. The standard generator α_i, $i = 1, \dots, \mu - 1$, acts as follows. Let $(\delta_1, \dots, \delta_\mu)$ be a distinguished basis of vanishing cycles and let s_δ denote the reflection corresponding to δ defined by $s_\delta(x) = x + \langle x, \delta \rangle \delta$. The action of α_i is given by

$$\alpha_i : (\delta_1, \dots, \delta_\mu) \mapsto (\delta'_1, \dots, \delta'_\mu) = (\delta_1, \dots, \delta_{i-1}, s_{\delta_i}(\delta_{i+1}), \delta_i, \delta_{i+2}, \dots, \delta_\mu).$$

We have $\langle \delta'_k, \delta'_j \rangle = \langle \delta_k, \delta_j \rangle$ for all $1 \leq k, j \leq \mu$, $k, j \neq i, i+1$, and

$$\begin{aligned}
\langle \delta'_i, \delta'_{i+1} \rangle &= -\langle \delta_i, \delta_{i+1} \rangle, \\
\langle \delta'_k, \delta'_{i+1} \rangle &= \langle \delta_k, \delta_i \rangle, \\
\langle \delta'_k, \delta'_i \rangle &= \langle \delta_k, \delta_{i+1} \rangle + \langle \delta_i, \delta_{i+1} \rangle \langle \delta_k, \delta_i \rangle,
\end{aligned}$$

where $k \neq i, i+1$. The inverse operation α_i^{-1} is also denoted by β_{i+1} and is given by

$$\beta_{i+1} : (\delta_1, \dots, \delta_\mu) \mapsto (\delta_1, \dots, \delta_{i-1}, \delta_{i+1}, s_{\delta_{i+1}}(\delta_i), \delta_{i+2}, \dots, \delta_\mu).$$