Topology and its Applications 234 (2018) 298–310

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Virtual Special Issue – Real and Complex Singularities and their applications in Geometry and Topology

Graphs of stable maps from closed surfaces to the projective plane

C. Mendes de Jesus [a,*,1], M.C. Romero-Fuster [b,2]

[a] *Departamento de Matemática, Universidade Federal de Viçosa, Brazil*
[b] *Departament de Matemàtiques, Universitat de València, Spain*

A R T I C L E   I N F O

Article history:
Received 30 January 2017
Received in revised form 10 July 2017
Accepted 18 July 2017
Available online 23 November 2017

MSC:
57R45
57R65
37E25

Keywords:
Closed surface
Stable map
Singular set
Global classification
Graphs

A B S T R A C T

We describe how to attach a weighted graph to each stable map from closed surfaces to projective plane and prove that any weighted graph with non negatively weighted vertices is the graph of some stable map from a closed surface to the projective plane.

1. Introduction

Stable maps between surfaces can only have fold curves with isolated cusp points on them (H. Whitney [2]). The global description of a stable map between closed surfaces requires the determination of both, the topological type of its regular set in the domain and the isotopy type of the image of its singular set (branch set or apparent contour) in the range surface. For this purpose, we introduced in [3] a graph with weights in its vertices that codifies the topological type of the regular set of stable maps from oriented surfaces to the plane. In subsequent papers [4–7] we studied the properties of such graphs and their behavior through surgeries and isotopies of maps from a closed orientable surface to the plane and to the sphere. We proved that any bipartite weighted graph can be the graph of a stable map from a closed surface to

* Corresponding author.
E-mail addresses: cmendes@ufv.br (C. Mendes de Jesus), carmen.romero@uv.es (M.C. Romero-Fuster).
1 Work partially supported by CNPq (200921/2012) and FAPEMIG (APQ-00715-14).
2 Work partially supported by DGCYT-FEDER grant n° MTM2015-64013-P.

https://doi.org/10.1016/j.topol.2017.11.013
0166-8641/

the plane (and hence to the sphere). Here we recall that a bipartite graph is a graph whose vertices can be divided into two disjoint sets such that every edge connects a vertex in one of them to one in the other. We also characterized those graphs that can be associated to stable maps without cusps (i.e. fold maps) from closed surfaces to the sphere, for each fixed degree d.

Our aim in this paper is to extend these results to the non orientable case. We shall thus consider stable maps from closed (non necessarily oriented) surfaces to the projective plane, extending the class of graphs previously defined in a way that enables us to distinguish among orientable and non-orientable regions in the complement of the singular set. We shall prove that any one of these graphs can be attached to a stable map from a (non necessarily oriented) surface to the projective plane.

One of the main tools used in this proof are the horizontal and vertical surgeries between stable maps introduced in [5]. These surgeries induce in turn some operations between their corresponding graphs. The strategy of our proof is the following: We start from basic graphs (irreducible trees and loops with just one vertex) and see how to attach them stable maps (Lemmas 4.2, 4.4 and 4.5). Then by an inductive process, based in the convenient manipulation of surgeries, we extend this results to any graph (Propositions 4.3 and 4.6 and Theorem 4.7). Some of these results are based in the existence of concrete examples of maps realizing a given basic graph. Such maps are qualitatively described through convenient figures along the paper. With the purpose of properly understanding the action of such maps, we represent them as the result of a factorization given by the composition of a convenient immersion of the surface in 3-space followed by a projection onto the real plane and an inclusion of the real plane into the projective plane.

The particular case of fold maps will be analyzed in a forthcoming paper.

2. Graphs of stable maps from closed surfaces to the projective plane

Given two closed surfaces M and N with respective genus g_M and g_N let $f : M \to N$ be a stable map between them. The singular set Σf is a finite collection of closed regular simple curves on M made of folds points with possible isolated cusp points. Moreover the image of Σf, known as the *apparent contour* or *branch set* of f, is a collection of closed curves in N with normal crossings and isolated singularities corresponding to the cusp points of f. Topological information of the map f may be conveniently encoded in a weighted graph from which the pair $M, \Sigma f$ may be reconstructed (up to diffeomorphism) ([4], [5]). The edges and vertices of this graph correspond (respectively) to the singular curves and the connected components of the non-singular set. An edge is incident to a vertex if and only if the singular curve corresponding to the edge lies in the frontier of the regular region corresponding to the vertex. In other words, given a stable map $f : M \to N$, its graph $\mathcal{G}(f)$ is the *dual graph* of Σf in M. The weight g_v of a vertex v is defined to be the genus of the corresponding region i.e the genus of the closed surface obtained by glueing a disk along each boundary curve of the region. Clearly, the graph of a stable map is an $\mathcal{A}$-invariant.

Our purpose in this paper is to extend the definition of graph to the class of stable maps from closed (non necessarily oriented) surfaces to the projective plane $\mathbb{P}$ and generalize the already known results on stable maps from closed orientable surfaces to the plane and the sphere.

We first observe that when the domain surface M is non orientable, the graph does no need to be bipartite. Moreover, we have two possibilities for a closed curve γ in M (Fig. 1):

a) The curve γ has a small neighborhood homeomorphic to a cylinder.
b) The curve γ has a small neighborhood homeomorphic to a Mœbius band.

According to this, for a stable map $f : M \to \mathbb{P}$ of a closed surface, we shall attach a $\star$ to each edge of the graph of f that corresponds to a singular curve of f having a neighborhood homeomorphic to a Mœbius band. Notice that such an edge always determines a loop, i.e. both ends correspond to a unique vertex of the graph. Fig. 1 shows the graph and the apparent contour of a stable map with a unique cusp from