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Exceptional rays and bilipschitz geometry of real surface singularities

Donal O'Shea [a], Leslie Wilson [b,*]

[a] *President's Office, New College of Florida, 5800 Bayshore Road, Sarasota, FL 34243, USA*
[b] *Mathematics Department, University of Hawai'i at Manoa, 2565 McCarthy Mall, Honolulu, HI 96822, USA*

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ABSTRACT

It is known that ambient bilipschitz equivalence preserves tangent cones. This paper explores the behavior of the Nash cone and, in particular, exceptional rays under ambient bilipschitz equivalence for real surfaces in $\mathbb{R}^3$ with isolated singularity.

1. Introduction

In [9], we extended work of Whitney [12], Lê [7], Teissier [8], and others [6] on limits of tangent spaces in the complex analytic setting to the case of real surfaces in $\mathbb{R}^3$. In recent years, there has been much progress on bilipschitz geometry for complex analytic surfaces (see, for example, [3], [4]), and it is again natural to ask whether, and how, results in the complex analytic case carry over to the reals.

To be more precise, and to fix notation, we let V be a semialgebraic surface in $\mathbb{R}^3$ containing the origin $\mathbf{0}$ (although all our results are stated in the semialgebraic category, they should be true in the subanalytic category as well). Two natural semialgebraic sets, the (Zariski) tangent cone, and the Nash cone, reflect the local geometry of V at $\mathbf{0}$. The tangent cone, $C \equiv CV \equiv C^+(V, \mathbf{0})$, denotes the set of tangent vectors: that is, $\mathbf{v} \in C$ if and only if there exist $\mathbf{x}_n \in V - \{\mathbf{0}\}, \mathbf{x}_n \to \mathbf{0}$ and a sequence of positive real numbers

* Corresponding author.
E-mail addresses: doshea@ncf.edu (D. O'Shea), les@math.hawaii.edu (L. Wilson).

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$t_n > 0$ such that $t_n\mathbf{x}_n \to \mathbf{v}$. The Nash cone, $\mathcal{N} \equiv \mathcal{N}V \equiv \mathcal{N}(V, \mathbf{0})$ denotes the set of 2-planes T with the property that there exists a sequence of $\mathbf{x}_n$ of smooth points of V (by which we mean points where V is locally a 2-dimensional C^1 manifold) converging to $\mathbf{0}$ such that T is the limit of tangent spaces to V at the points $\mathbf{x}_n$. By passing to a subsequence if necessary, we can assume that the sequence $\{\mathbf{x}_n\}$ approaches the origin tangent to some ray ℓ. Necessarily, $\ell \subset C$. We let $\mathcal{N}_\ell(V, \mathbf{0}) \subset \mathcal{N}(V, \mathbf{0})$ denote the space of limits of tangent spaces that can be obtained as limits of tangent spaces along sequences tending to the origin tangent to ℓ. Whitney shows that if V is algebraic, then $T \in \mathcal{N}_\ell$ implies $\ell \subset T$ (a result that extends easily to the case V semialgebraic).

In [9], we establish the analog of the Lê–Teissier theorem for algebraic surfaces $V \subset \mathbb{R}^3$ containing the origin $\mathbf{0}$; that is, for surfaces given implicitly by an equation $\{f = 0\}$ where $f \in \mathbb{R}[x, y, z]$ is a polynomial vanishing at $\mathbf{0}$. However, the techniques and results of [9] apply to semialgebraic surfaces in $\mathbb{R}^3$. In particular, we show that if $V \subset \mathbb{R}^3$ is a reduced, semialgebraic surface with $\mathbf{0}$ an isolated singularity, then there exist finitely many rays $\ell_1, \ldots, \ell_r$ in C, called exceptional rays, with $\mathcal{N}_{\ell_i}$ connected, closed and one-dimensional. For any other ray $\ell \in C - \{\ell_1, \ldots, \ell_r\}$, $\mathcal{N}_\ell(V, \mathbf{0})$ is a single point (that is a single plane), and $\mathcal{N}_\ell(V, \mathbf{0}) = \mathcal{N}_\ell(C, \mathbf{0})$. An exceptional ray ℓ is said to be *full* if $\mathcal{N}_\ell$ consists of the full pencil of planes in $\mathbb{R}^3$ containing ℓ. In the case of complex analytic surfaces, all exceptional lines are full, so that knowledge of the tangent cone and exceptional rays completely characterizes the Nash cone, so that Lê–Teissier's work [8] together with that of Birbrair, Neumann and Pichon [4] allows one to sketch out the basics of a theory of bilipschitz geometry for complex surfaces. What of real surfaces?

2. Exceptional rays necessitated by the topology

A map $h : V \to W$ between two metric spaces (V, d_V) and (W, d_W) is said to be *lipschitz* if

$$d_W(h(x), h(y)) \le K d_V(x, y)$$

for all $x, y \in V$ and some constant $K > 0$, and *bilipschitz* if h^{-1} exists and is lipschitz. Equivalently, $h : V \to W$ is bilipschitz if and only if h is surjective and there exists $K > 0$ such that

$$\frac{1}{K} d_V(x, y) \le d_W(h(x), h(y)) \le K d_V(x, y).$$

A semialgebraic set V, real or complex, embedded in $\mathbb{R}^n$ or $\mathbb{C}^n$ has two natural metrics. One, the *intrinsic* or *inner* metric on V is the metric induced on V by defining the distance $d_i(x, y)$ between two points x and y to be the infimum of the lengths of piecewise analytic arcs on V joining x and y. The *outer* metric on V defines the distance between any two points x and y to be their Euclidean distance $d_o(x, y) = |x - y|$ in the ambient space. Two such sets V, W will be said to be inner (resp. outer) bilipschitz homeomorphic if they are bilipschitz homeomorphic with respect to the inner (resp. outer) metrics. If we don't say otherwise, we will mean outer. In addition, if the outer bilipschitz homeomorphism is the restriction of a bilipschitz homeomorphism on a neighborhood of the sets in Euclidean space, we will say they are ambient bilipschitz homeomorphic. A homeomorphism is semialgebraic if its graph is semialgebraic. All our bilipschitz homeomorphisms are assumed to preserve the origin.

In [10], Sampaio shows that two semialgebraic sets that are outer bilipschitz homeomorphic have outer bilipschitz homeomorphic tangent cones. Although it is no longer quite true over the reals that exceptional rays together with the tangent cone completely characterize the Nash cone, the exceptional rays play an important role, and it is natural to ask whether bilipschitz homeomorphic semialgebraic sets have the same exceptional rays up to bilipschitz equivalence. We shall see shortly that this is not the case. Nonetheless, there are instances in which (see [9]) the topology of the tangent cone forces the existence of exceptional rays. In such cases, two semialgebraic surfaces which are bilipschitz equivalent necessarily have exceptional rays. Three cases are worth singling out. All surfaces are understood to be in $\mathbb{R}^3$.