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On discriminants, Tjurina modifications and the geometry of determinantal singularities

Anne Frühbis-Krüger [1]

Institut für algebraische Geometrie, Leibniz Universität Hannover, 30167 Hannover, Germany

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We describe a method for computing discriminants for a large class of families of isolated determinantal singularities – families induced by perturbations of matrices. The approach intrinsically provides a decomposition of the discriminant into two parts and allows the computation of the determinantal and the non-determinantal loci of the family without extra effort; only the latter manifests itself in the Tjurina transform. This knowledge is then applied to the case of Cohen–Macaulay codimension 2 singularities putting several known, but previously unexplained observations into context and explicitly constructing a counterexample to Wahl's conjecture (see [35], section 6) on the relation of Milnor and Tjurina numbers for surface singularities.

1. Introduction

Isolated hypersurface and complete intersection singularities are well studied objects and there are many classical results about different aspects such as topology, deformation behaviour, invariants, classification and even metric properties (see any textbook on singularities, e.g. [25], [22], [24]). Beyond complete intersections, however, knowledge is rather scarce and unexpected phenomena arise. In this article, we focus on the class of determinantal singularities to pass beyond ICIS, as the properties already differ significantly, but classical results on determinantal varieties and free resolutions provide strong tools to treat this case. Recently, significant progress has been made for this class, e.g. in [30], [3], [28], [11], [19]. In [18] the use of Tjurina

E-mail address: anne@math.uni-hannover.de.

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modifications made it possible to relate a given determinantal singularity to an often singular variety, which happens to be an ICIS under rather mild conditions. This could e.g. be exploited in [18] and [36] to determine the topology of the Milnor fibre of an isolated Cohen–Macaulay codimension 2 singularity (ICMC2 singularity for short). But it is also obvious from those results that the Tjurina transform is blind to certain other properties of an ICMC2 singularity. In this article, we explain which properties of the singularity manifest themselves in the Tjurina transform and which do not by studying the discriminant and a natural decomposition thereof.

The general approach to determining the discriminant of a given family of varieties $V(I_{\underline{t}})$ is based on the Jacobian criterion. It involves the elimination of the original variables from the ideal generated by $I_{\underline{t}}$ and the ideal of minors of appropriate size of the relative Jacobian matrix of $I_{\underline{t}}$. However, the complexity of this approach, which originates from the sensitivity of Gröbner basis computations to the number of occurring variables, makes it impractical for many examples. It is hence important to understand the structure of the discriminant theoretically and to be able to decompose it appropriately by a priori arguments. Making use of Hironaka's smoothness criterion[2] the structure of the perturbed matrix can be used to split the problem into two smaller problems, one dealing with the locus of determinantal singularities, the other one closely related to the Tjurina transform as it describes the locus above which there are singularities adjacent to an A_1 singularity.

In section 2, we first recall known facts about determinantal singularities and then proceed to revisit Hironaka's smoothness criterion. In the following section 3, we consider the discriminant of versal families of determinantal singularities of type (m, n, t) starting with the simplest case $(2, k, 2)$, then passing on to maximal minors of matrices of arbitrary size and finally to smaller minors. As a side-effect, we also obtain a quite explicit formulation of the Tjurina transform in the non-maximal case. The above mentioned decomposition of the discriminant into the two parts also gives rise to the surprising behaviour of some aspects of ICMC2 singularities, as we are seeing an interplay of influences related to properties of the generic determinantal singularity and to the Tjurina transform. With these two contributions in mind, it is possible to predict some properties of the singular locus of the Tjurina transform, to explain observations of [11] about ICMC2 3-folds and prove the easy direction of Wahl's conjecture [35] on the relation between Milnor and Tjurina number for ICMC2 surface singularities. The knowledge from this proof then leads to the construction of a class of counter examples for the converse direction of the conjecture. These applications to the ICMC2 case are discussed in the final section.

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2. Basic facts on EIDS

Before focussing on the discriminant, we shall first recall the definition and properties of the class of singularities on which we focus in the following sections: essentially isolated determinantal singularities (EIDS) as first introduced by Ebeling and Gusein-Zade in [13]. Although certain classes of such singularities

[2] Cf. [23], where this criterion is hidden in the fact that the invariant ν^* (defined in Chapter III, Definition 1), which is specifically designed to decrease in the resolution process, attains its minimal value precisely at the smooth points; this has recently been exploited computationally as a parallel smoothness test in [4], section 2.