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Relative equivariants under compact Lie groups

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ABSTRACT

In this work we obtain the general form of polynomial mappings that commute with a linear action of a relative symmetry group. The aim is to give results for relative equivariant polynomials that correspond to the results for relative invariants obtained in a previous paper (Baptistelli and Manoel (2013) [5]). We present an algorithm to compute generators for relative equivariant submodules from the invariant theory applied to the subgroup formed only by the symmetries. The same method provides, as a particular case, generators for equivariants under the whole group from the knowledge of equivariant generators by a smaller subgroup, which is normal of finite index.

1. Introduction

Occurrence of symmetries provides a powerful and useful tool in the formulation and analysis of a model. The appropriate combination of the geometry of a configuration space with its inherent symmetries can show, in a natural way, existence of many phenomena not expected when symmetries are not present. This is due to the fact that they are the reason for common observations such as degeneracy of solutions, high codimension bifurcations, unexpected stabilities, phase relations, synchrony, periodicity of solutions, among many others. The set of symmetries appears in the model with different algebraic structures in different contexts. In particular, the study of dynamical systems has been greatly developed throughout many years when the collection of symmetries has a group structure. Group representation theory is in

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this case a systematic way that enables us to deal with this question in distinct aspects in applications. More specifically, in the numerical analysis of such systems through computational programming as, for example, in [10,11,16]; also, in algebraic methods that provide the general form of the vector fields through the description of invariant functions and commuting mappings under distinct actions of symmetry groups as in [1,2,5]. In addition, also based on group representation are the analysis of steady-state bifurcation, Hopf bifurcation, periodic cycles, classification of singularities and so on, for which we cite for example [3,4,6,13] and the books [8,12] with hundreds of references therein. More recently, in a related but distinct direction, many authors have investigated the occurrence of symmetries in networks of dynamical systems. Networks are schematically represented by a graph, and symmetries enter in a more subtle way, even when the architecture of the graph presents no symmetry. The algebraic structure in this case is of a groupoid of symmetries. This has been first formalized in [15] and has been successfully applied since then by many authors.

The results in the present work are related to symmetries in differential operations on complex variables. We obtain a procedure from invariant theory to give the general form of mappings that commute with a complex linear action of a group Γ, the group of relative symmetries. We assume that Γ is a compact Lie group whose subgroup of symmetries is a normal subgroup K of finite index m greater than two. More specifically, we consider a group epimorphism

$$\sigma : \Gamma \to \mathbf{Z}_m, \tag{1}$$

where $\mathbf{Z}_m$ denotes the cyclic group generated by the m-th root of unity and $K = \ker \sigma$. The case $m = 2$ corresponds to reversible equivariant mappings and it has been studied in [1]. It models vector fields whose integral curves come into two types of symmetric families: the time-preserving symmetric family, each element of which given from an element of K (the subgroup of symmetries) and the time-reversing symmetric family, whose elements come from each element of the complement $\Gamma \setminus K$ (the subset of reversible symmetries). We point out that in another direction one might consider an epimorphism $\Gamma \to \mathbf{S}^1$. This has not been addressed to our attention yet and, although it seems to be an interesting problem, it may require a distinct machinery than those used for the cyclic group $\mathbb{Z}_m$.

Here we treat the case $m > 2$. We generalize previous results done for $m = 2$ to the cases $m > 2$, m finite. The $m = 2$ case is related to reversible equivariant dynamics, once the epimorphism $\sigma : \Gamma \to \mathbf{Z}_2$ in the equation will imply that a symmetry $\gamma \in \Gamma$ will take trajectories into trajectories either preserving time – if $\sigma(\gamma) = 1$ – or reverting time – if $\sigma(\gamma) = -1$. Potential applications in dynamical systems of the corresponding situation when $m > 2$ and finite should consider the variable corresponding to time to be complex. These are related to differential equations in complex domain and to the theory of relativity in complex time, which not only complies with the possibility of time decomposition into two dimensions, but also conciliates with the idea of a complex space (see [9,14]).

From now on we assume throughout that Γ is a compact Lie group that admits an index-m normal subgroup K, the kernel of (1), and $\delta \in \Gamma$ is such that $\sigma(\delta)$ is the primitive m-th root of unity. We then have the decomposition of Γ as a disjoint union of left-cosets:

$$\Gamma = \dot{\bigcup}_{j=0}^{m-1} \delta^j K. \tag{2}$$

The results in [5] regard relative invariants. In this paper, we derive the corresponding algebraic results for relative equivariants.

For a linear action of Γ on a finite-dimensional vector space V over $\mathbf{C}$, consider the ring $\mathcal{P}(\Gamma)$ of the Γ-invariant polynomials, namely the ring of polynomial functions $f : V \to \mathbf{C}$ such that $f(\gamma x) = f(x)$, for all $\gamma \in \Gamma$ and $x \in V$. For each $j \in \{1, \ldots, m-1\}$, a polynomial function $f : V \to \mathbf{C}$ is called σ^j*-relative invariant* if

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