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Dynamics and limiting behavior of Julia sets of König's method for multiple roots ☆

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ABSTRACT

A well known result of J. Hubbard, D. Schleicher and S. Sutherland (see [27]) shows that if f is a complex polynomial of degree d, then there is a finite set S_d depending only on d such that, given any root α of f, there exists at least one point in S_d converging under iterations of N_f to α. Their proof depends heavily on the simply connectedness of the immediate basins of attraction of Newton's method. We show that for all order $\sigma \geq 2$, there exists a complex polynomial f such that the Julia set of König's method for multiple roots applied to it is disconnected. Consequently, our result establishes restrictions for extending the main result in [27] to higher order root-finding methods. As far as we know, there are no pictures of disconnected Julia sets for root finding algorithms applied to polynomials. Here we give a proof and provide pictures that illustrate such disconnectedness. We also show that the Fatou set of König's method for multiple roots converges to the Voronoi diagram under order of convergence growth, in the Hausdorff complementary metric.

1. Introduction

Newton's iterative method

$$N_f = Id - \frac{f}{f'} \tag{1.1}$$

where f is a complex polynomial, as well as other higher-order methods have been extensively studied and used in order to deal with the equation $f(z) = 0$ (see for example [1], [17], [32], [33], [36], and [49]). The iterative function N_f defines a rational map on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The simple roots of the equation $f(z) = 0$, that is the roots of the equation $f(z) = 0$ which are not roots of the derivative $f'(z)$, are super-attracting fixed points of N_f. In other words, let α be a simple root of $f(z)$. Then $N_f(\alpha) = \alpha$ and $N'_f(\alpha) = 0$. For a review of the dynamics of Newton's method, see for example [11], [24]. In a more

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general setting, we let $Poly_d$ and Rat_k denote the space of polynomials of degree d and the space of rational functions of degree k, respectively. A *root-finding algorithm* is a rational map $T_f : Poly_d \to Rat_k$ such that the roots of the polynomial map f are attracting fixed points of T_f. We will say that a root-finding algorithm T_f has *order of convergence* σ if the local degree of T_f at each simple root of f is σ.

E. Schröder ([44], [45], 1870) showed that Newton's method applied to quadratic polynomials in the complex plane has two basins of attraction associated with the corresponding roots. His proof is obtained by conjugating Newton's method for a quadratic polynomial f with the so-called *Newton's method for multiple roots* associated to f defined by

$$M_f(z) = z - \frac{f(z)f'(z)}{(f'(z))^2 - f(z)f''(z)}\,. \tag{1.2}$$

In fact, for instance conjugating Newton's method with a Möbius transformation sending one root to infinity and the other one to 0, we obtain the map z^2 which proves immediately the existence of two connected basins of attraction. More precisely, the basins of attraction associated to Newton's method applied to the polynomial $f(z) = z^2 - 1$ are found explicitly by noting that

$$N_f = \frac{1}{M_f}.$$

Thus, it suffices to concentrate on the convergence of the iterates under M_f. Consequently, the points on the complex plane with positive real part converge to 1, while those with negative real part converge to -1. For a different approach, see A. Cayley ([14], 1890). For a historical background and detailed explanation of these facts, see [3], [6]. More recently, Newton's method for multiple roots has also been studied by T. Pomentale [38] and W. Gilbert [22].

A natural generalization of Newton's method (1.1) is König's root-finding algorithm applied to polynomials, denoted $K_{f,\sigma}$, also known as the Basic Family (in short, König's method). Let $f : \mathbb{C} \to \mathbb{C}$ be a meromorphic map, and let $\sigma \geq 2$ be an integer. *König's method of order σ associated to f* is the meromorphic map $K_{f,\sigma} : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by

$$K_{f,\sigma} = Id + (\sigma - 1)\frac{(1/f)^{[\sigma-2]}}{(1/f)^{[\sigma-1]}} \tag{1.3}$$

where $(1/f)^{[k]}$ is the k^{th} derivative of $1/f$. In the special cases $\sigma = 2$ and $\sigma = 3$, König's method is Newton's method and Halley's method, respectively.

König's method applied to polynomials has been studied from both the numerical and the dynamical points of view. (See for example, [4], [13], [28], [29], [37], [50].)

Here we are interested in studying dynamical and geometric aspects of König's method for multiple roots (see Definition 2.1), denoted $M_{f,\sigma}$, which is a generalization of Newton's method for multiple roots. Given a polynomial f, König's method for multiple roots $M_{f,\sigma}$ is none other than König's method $K_{f,\sigma}$ applied to the rational map f/f'. Note that the term f/f' has the effect of converting the multiple roots of $f(z)$ into simple ones. In other words, if f has a multiple zero of order n at α, then f/f' has simple zero at α.

Although $K_{f,\sigma}$ and $M_{f,\sigma}$ are similar, they are different from a dynamical viewpoint. For example, infinity is never a fixed point for $M_{f,\sigma}$, which is exactly the opposite for $K_{f,\sigma}$. In addition, the fixed points of $M_{f,\sigma}$ associated to the roots of f are always super-attracting, regardless their multiplicity.

We next state our two main results.

In [13], X. Buff and C. Henriksen show that the sequence of Julia sets of König's method applied to a polynomial f converges to the union of the bisecting locus of the set of roots of the polynomial f and infinity as $\sigma \to \infty$. A Voronoi cell of a root α is the set of points in the complex plane that are strictly closer to α