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Embedding into free topological vector spaces on compact metrizable spaces

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ABSTRACT

For a Tychonoff space X, let $\mathbb{V}(X)$ be the free topological vector space over X. Denote by $\mathbb{I}$, $\mathbb{G}$, Q and $\mathbb{S}^k$ the closed unit interval, the Cantor space, the Hilbert cube $Q = \mathbb{I}^{\mathbb{N}}$ and the k-dimensional unit sphere for $k \in \mathbb{N}$, respectively. The main result is that $\mathbb{V}(\mathbb{R})$ can be embedded as a topological vector space in $\mathbb{V}(\mathbb{I})$. It is also shown that for a compact Hausdorff space K: (1) $\mathbb{V}(K)$ can be embedded in $\mathbb{V}(\mathbb{G})$ if and only if K is zero-dimensional and metrizable; (2) $\mathbb{V}(K)$ can be embedded in $\mathbb{V}(Q)$ if and only if K is metrizable; (3) $\mathbb{V}(\mathbb{S}^k)$ can be embedded in $\mathbb{V}(\mathbb{I}^k)$; (4) $\mathbb{V}(K)$ can be embedded in $\mathbb{V}(\mathbb{I})$ implies that K is finite-dimensional and metrizable.

1. Introduction

In [6] we introduced the study of free topological vector spaces and showed that, in some ways, the topological structure of free topological vector spaces is nicer and better understood than that of free locally convex spaces; in particular, the free topological vector space on a k_ω-space is a k_ω-space and

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therefore complete. By contrast the free locally convex space on an infinite space X is complete if and only if X is Dieudonné complete and has no infinite compact subsets, see [14].

For many years, [1,7,8,10,12–14], the following question has been investigated: for what Tychonoff spaces X and Y can the free (free abelian) topological group $F(Y)$ (respectively, $A(Y)$) on Y be embedded as a topological subgroup of the free (respectively, free abelian) topological group $F(X)$ (respectively, $A(X)$) on X? An analogous question for free locally convex spaces $L(X)$ and $L(Y)$ over X and Y respectively was considered in [11]. We note that, in general, the question for free abelian topological groups is harder than that for free topological groups and the question for free locally convex spaces is harder again. In this paper we investigate the following:

Question. For what Tychonoff spaces X and Y can the free topological vector space $\mathbb{V}(Y)$ be embedded as a topological vector subspace of the free topological vector space $\mathbb{V}(X)$?

The *free topological vector space* $\mathbb{V}(X)$ over a Tychonoff space X is a pair consisting of a topological vector space $\mathbb{V}(X)$ and a continuous map $i = i_X : X \to \mathbb{V}(X)$ such that every continuous map f from X to a topological vector space E gives rise to a unique continuous linear operator $\bar{f} : \mathbb{V}(X) \to E$ with $f = \bar{f} \circ i$. Theorem 2.3 of [6] shows that for all Tychonoff spaces X, $\mathbb{V}(X)$ exists, is unique up to isomorphism of topological vector spaces, is Hausdorff and the mapping i is a homeomorphism of the topological space X onto its image in $\mathbb{V}(X)$. For a subspace Y of X, let $\mathbb{V}(Y, X)$ be the vector subspace of $\mathbb{V}(X)$ spanned by Y. Concerning the question raised above, Proposition 3.12 of [6] shows that, if K is a compact subspace of the Tychonoff space X, then the vector subspace $\mathbb{V}(K, X)$ of $\mathbb{V}(X)$ is $\mathbb{V}(K)$. Further, Corollary 3.11 of [6] says that if Y is a closed subspace of the k_ω-space X, then the vector subspace $\mathbb{V}(Y, X)$ of $\mathbb{V}(X)$ is $\mathbb{V}(Y)$.

It is known [9] that the free topological group $F(\mathbb{R})$ on $\mathbb{R}$ can be embedded as a topological group in the free topological group $F(\mathbb{I})$ on the unit interval $\mathbb{I}$. It is also known [8] that the free abelian topological group $A(\mathbb{R})$ on $\mathbb{R}$ can be embedded as a topological group in the free abelian topological group $A(\mathbb{I})$ on the unit interval $\mathbb{I}$, but this is significantly harder to prove. We hasten to point out that while $\mathbb{R}$ can be embedded as a subspace of $\mathbb{I}$, no subgroup of $F(\mathbb{I})$ or $A(\mathbb{I})$ generated by a subspace Y of $\mathbb{I}$ is isomorphic as a topological group to $F(\mathbb{R})$ or $A(\mathbb{R})$. It is shown in Theorem 4.3 of [11] that the free locally convex space $L(\mathbb{R})$ on $\mathbb{R}$ cannot be embedded as a topological vector subspace of the free locally convex space $L(\mathbb{I})$ on $\mathbb{I}$.

The main theorem in this paper is that the free topological vector space $\mathbb{V}(\mathbb{R})$ on $\mathbb{R}$ can indeed be embedded as a topological vector space in the free topological vector space $\mathbb{V}(\mathbb{I})$ on $\mathbb{I}$.

It is also shown that for a compact Hausdorff space K:

(1) $\mathbb{V}(K)$ can be embedded in $\mathbb{V}(\mathbb{G})$, where $\mathbb{G}$ is the Cantor space, if and only if K is zero-dimensional and metrizable;
(2) $\mathbb{V}(K)$ can be embedded in $\mathbb{V}(Q)$, where Q denotes the Hilbert cube $Q = \mathbb{I}^{\mathbb{N}}$, if and only if K is metrizable;
(3) $\mathbb{V}(\mathbb{S}^k)$ can be embedded in $V(\mathbb{I}^k)$, where $\mathbb{S}^k$ denotes the k-dimensional sphere, $k \in \mathbb{N}$;
(4) $\mathbb{V}(K)$ can be embedded in $\mathbb{V}(\mathbb{I})$ implies that K is finite-dimensional and metrizable.

2. Universal spaces

In this section we study free topological vector spaces generated by the universal spaces: (1) the Hilbert cube $Q = \mathbb{I}^{\mathbb{N}}$; it is well-known that every metrizable compact space embeds into Q; (2) the Cantor compact space $\mathbb{G}$; it is known that every zero-dimensional compact metrizable space embeds into $\mathbb{G}$, see Example 3.1.28 and Theorem 6.2.16 of [3]. In so doing, we obtain new results not only for free topological vector spaces, but also for free locally convex spaces on these universal spaces.

In what follows we use the following notation. Set $\mathbb{N} := \{1, 2, \dots\}$. For a subset A of a vector space E and a natural number $n \in \mathbb{N}$ we denote by $\mathrm{sp}_n(A)$ the following subset of E