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On dense subsets, convergent sequences and projections of Tychonoff products ☆

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ABSTRACT

It is well know that the Tychonoff product of 2^ω many separable spaces is separable [2,3].
We consider for the Tychonoff product of 2^ω many separable spaces the problem of the existence of a dense countable subset, which contains no nontrivial convergent in the product sequences.
The first result was proved by W.H. Priestley. He proved [14] that such dense set exists in the Tychonoff product $\prod_{\alpha\in 2^\omega} I_\alpha$ of closed unit intervals.
We prove (Theorem 3.2) that such dense set exists in the Tychonoff product $\prod_{\alpha\in 2^\omega} Z_\alpha$ of 2^ω many Hausdorff separable not single point spaces.
We prove that in $\prod_{\alpha\in 2^\omega} Z_\alpha$ there is a countable dense set $Q \subseteq \prod_{\alpha\in 2^\omega} Z_\alpha$ such that for every countable subset $S \subseteq Q$ a set $\pi_A(S)$ is dense in a face $\prod_{\alpha\in A} Z_\alpha$ for some A, $|A| = \omega$.
We prove (Theorem 3.4) that in $\prod_{\alpha\in 2^\omega} I_\alpha$ there is a countable set, that is dense but sequentially closed in $\prod_{\alpha\in 2^\omega} I_\alpha$ with the Tychonoff topology and is closed and discrete in $\prod_{\alpha\in 2^\omega} I_\alpha$ with the box topology (Theorem 3.4).

1. Introduction

By Hewitt–Marczewski–Pondiczery theorem (see [2,3]), the Tychonoff product $\prod_{\alpha\in 2^\omega} X_\alpha$ of 2^ω many separable spaces is separable.

We consider the problem of the existence in the Tychonoff product of 2^ω many separable spaces a dense countable subset, which contains no nontrivial convergent in the product sequences.

In [14] W.H. Priestley proved that such countable dense set exists in I^{2^ω}, where I is closed unit interval.

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In [15] P. Simon proved that such countable dense set exists in D^{2^ω}, where D is a two-point discrete space. He proved that in D^{2^ω} there is a countable dense set such that the closure of every countable subset of it has a cardinality 2^{2^ω}.

In [10] we proved that such countable dense set exists in Z^{2^ω}, where Z is a countable discrete space. We proved that in Z^{2^ω} there is a countable dense set such that every countable subset of Q contains a countable discrete in Z^{2^ω} subset.

Now we in Theorem 3.2 prove that such countable dense set exists in the general case of the product $\prod_{\alpha\in 2^\omega} Z_\alpha$ of not one-point Hausdorff separable spaces.

We prove that in $\prod_{\alpha\in 2^\omega} Z_\alpha$ there is a countable dense set $Q \subseteq \prod_{\alpha\in 2^\omega} Z_\alpha$ such that for every countable subset $S \subseteq Q$ a set $\pi_A(S)$ is dense in a face $\prod_{\alpha\in A} Z_\alpha$ for some A, $|A| = \omega$

We use the notion of the independent matrix (see Preliminaries), it generalizes the independent family of sets [5,11].

Independent families of sets, in particular, were used by R. Engelking and M. Karlowicz ([4]) for their proof of Hewitt – Marczewski – Pondiczery theorem, by P. Simon ([15]) in his theorem, mentioned above.

The notion of independent matrix was defined by J. van Mill [13] as a subfamily of independent linked family, defined by K. Kunen [12].

Families of this type were widely used in the theory of Stone–Chech compactifications of discrete spaces (see in particular [12,13,6,7]).

We used independent matrices in [8–10] for an investigation of dense subsets of products.

2. Preliminaries

Definitions and notions used in the paper can be found in [1–3]. $d(X)$ denotes the density of a space X, by $[A]$ we denote the closure of a set A, $\exp A$ denote the set of all subsets of A and by $\operatorname{Exp} A$ we denote the set of all non-empty subsets of a set A. By Y^X we denote the set of all mappings from X to Y. We say that X is a countable set if $|X| = \omega$.

By π_α we denote an α-projection of $\prod_{\alpha\in A} X_\alpha$ on X_α.

A sequence $\{x_n\}_{n=1}^{\infty}$ is called trivial if there is $n_0 \in \omega$ such that $x_n = x_{n_0}$ for all $n \geq n_0$.

A subset $A \subseteq X$ is called sequentially closed if A contains limits of all its convergent in X sequences.

We will use the notion of the independent matrix of subsets of ω (see [13]).

Definition 2.1. ([13]) An indexed family $\{A_{ij} : i \in I, j \in J\}$ of subset of ω is called a J by I independent matrix if

- whenever j_0, $j_1 \in J$ are distinct and $i \in I$, then $|A_{ij_0} \cap A_{ij_1}| < \omega$;
- if $i_1, \dots, i_n \in I$ are distinct and $j_1, \dots, j_n \in J$, then

$$|\cap\{A_{i_k j_k} : k = 1, \dots, n\}| = \omega.$$

For $\{A_{ij} : i \in I, j \in J\}$ the family $\{A^*_{ij} = [A_{ij}] \setminus \omega : i \in I, j \in J\}$ is a J by I independent matrix of clopen subsets of ω^*.

The following construction of the 2^ω by 2^ω independent matrix of subsets of ω can be found in [13].

Let

$$H' = \{< k, u >: k \in \omega, u \in (\exp k)^{\exp k}\}$$