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Lexicographic products of GO-spaces

Nobuyuki Kemoto

Department of Mathematics, Oita University, Oita, 870-1192 Japan

A R T I C L E I N F O

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It is known that lexicographic products of paracompact LOTS's are also paracompact, see [2]. In this paper, the notion of lexicographic products of GO-spaces is defined. We characterize when a lexicographic product of GO-spaces is a LOTS. Moreover, we show that lexicographic products of paracompact GO-spaces are also paracompact. For example, we see

- the lexicographic products $\mathbb{M} \times \mathbb{P}$ and $\mathbb{S} \times [0,1)_{\mathbb{R}}$ are LOTS's, but $\mathbb{P} \times \mathbb{M}$ and $\mathbb{S} \times (0,1]_{\mathbb{R}}$ are not LOTS's,
- the lexicographic product $\mathbb{S}^\gamma$ of the γ-many copies of $\mathbb{S}$ is a LOTS iff γ is a limit ordinal,
- the lexicographic products $\mathbb{M} \times \mathbb{P}$ and $\mathbb{P} \times \mathbb{M}$ are paracompact,
- the lexicographic product $\mathbb{S}^\gamma$ is paracompact for every ordinal γ,

where $\mathbb{P}$, $\mathbb{M}$, $\mathbb{S}$ and $[0,1)_{\mathbb{R}}$ denote the irrationals, the Michael line, the Sorgenfrey line and the interval $[0,1)$ in the reals $\mathbb{R}$, respectively.

1. Introduction

We assume all topological spaces have cardinality at least 2.

A linearly ordered set $\langle X, <_X \rangle$ (see [1]) has a natural T_2-topology denoted by λ_X or $\lambda(<_X)$ so called the *interval topology* which is the topology generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\}$ as a subbase, where $(x, \rightarrow)_X = \{w \in X : x <_X w\}$, $(x, y]_X = \{w \in X : x <_X w \leq_X y\}$, ..., etc. Here $w \leq_X x$ means $w <_X x$ or $w = x$. If the contexts are clear, we simply write $<$ and $(x, y]$ instead of $<_X$ and $(x, y]_X$ respectively. Note that this subbase induces a base by convex subsets (e.g., the collection of all intersections of at most two members of this subbase), where a subset B of X is *convex* if for every $x, y \in B$ with $x <_X y$, $[x, y]_X \subset B$ holds. The triple $\langle X, <_X, \lambda_X \rangle$ is called a *LOTS* (= Linearly Ordered Topological Space) and simply denoted by LOTS X. Observe that if $x \in U \in \lambda_X$ and $(\leftarrow, x) \neq \emptyset$, then there is $y \in X$ such that

E-mail address: nkemoto@cc.oita-u.ac.jp.

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$y < x$ and $(y, x] \subset U$. Note that for every $x \in X$, $(\leftarrow, x] \notin \lambda_X$ iff $(x, \rightarrow)$ is non-empty and has no minimum (briefly, min), also analogously $[x, \rightarrow) \notin \lambda_X$ iff $(\leftarrow, x)$ is non-empty and has no max. Let

$$X_R = \{x \in X : (\leftarrow, x] \notin \lambda_X\} \text{ and } X_L = \{x \in X : [x, \rightarrow) \notin \lambda_X\}.$$

Unless otherwise stated, the real line $\mathbb{R}$ is considered as a linearly ordered set (hence LOTS) with the usual order, similarly so are the set $\mathbb{Q}$ of rationals, the set $\mathbb{P}$ of irrationals and an ordinal α.

A *generalized ordered space* (= GO-space) is a triple $\langle X, <_X, \tau_X \rangle$, where $<_X$ is linear order on X and τ_X is a T_2 topology on X which has a base consisting of convex sets, also simply denoted by GO-space X. For LOTS's and GO-spaces, see also the nice text book [5]. It is easy to verify that τ_X is stronger than λ_X. Also let

$$X^+_{\tau_X} = \{x \in X : (\leftarrow, x]_X \in \tau_X \setminus \lambda_X\},$$
$$X^-_{\tau_X} = \{x \in X : [x, \rightarrow)_X \in \tau_X \setminus \lambda_X\}.$$

Obviously $X^+_{\tau_X} \subset X_R$ and $X^-_{\tau_X} \subset X_L$. When contexts are clear, we usually simply write X^+ and X^- instead of $X^+_{\tau_X}$ and $X^-_{\tau_X}$, respectively. Note that X is a LOTS iff $X^+ \cup X^- = \emptyset$. For $A \subset X_R$ and $B \subset X_L$, let $\tau(A, B)$ be the topology generated by $\{(\leftarrow, x)_X : x \in X\} \cup \{(x, \rightarrow)_X : x \in X\} \cup \{(\leftarrow, x]_X : x \in A\} \cup \{[x, \rightarrow)_X : x \in B\}$ as a subbase. Obviously $\tau_X = \tau(X^+, X^-)$ whenever X is a GO-space, and also $\tau(A, B)$ defines a GO-space topology on X whenever X is a LOTS with $A \subset X_R$ and $B \subset X_L$. The Sorgenfrey line $\mathbb{S}$ is $\langle \mathbb{R}, <_{\mathbb{R}}, \tau(\mathbb{R}, \emptyset) \rangle$ and the Michael line $\mathbb{M}$ is $\langle \mathbb{R}, <_{\mathbb{R}}, \tau(\mathbb{P}, \mathbb{P}) \rangle$. These spaces are GO-spaces but not LOTS's.

Let X be a GO-space and $Y \subset X$, then "the subspace Y of a GO-space X" means the GO-space $\langle Y, <_X \restriction Y, \lambda_X \restriction Y \rangle$, where $<_X \restriction Y$ is the restricted order of $<_X$ on Y and $\lambda_X \restriction Y := \{U \cap Y : U \in \lambda_X\}$, that is, $\lambda_X \restriction Y$ is the subspace topology of λ_X.

Now for a given GO-space X, let

$$X^* = \big(X^- \times \{-1\}\big) \cup \big(X \times \{0\}\big) \cup \big(X^+ \times \{1\}\big)$$

and consider the lexicographic order $<_{X^*}$ on X^* induced by the lexicographic order on $X \times \{-1, 0, 1\}$, here of course $-1 < 0 < 1$. We usually identify X as $X = X \times \{0\}$ in the obvious way (i.e., $x = \langle x, 0 \rangle$), thus we may consider $X^* = \big(X^- \times \{-1\}\big) \cup X \cup \big(X^+ \times \{1\}\big)$. Note $(\leftarrow, x]_X = (\leftarrow, \langle x, 1 \rangle)_{X^*} \cap X \in \lambda(<_{X^*}) \restriction X$ whenever $x \in X^+$, and also its analogy. Then the GO-space X is a dense subspace of the LOTS X^*, and X has max iff X^* has max, in this case, $\max X = \max X^*$ (and similarly for min). Note $\mathbb{S}^* = \mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\}$ with the identification $\mathbb{S} = \mathbb{R} \times \{0\}$ and $\mathbb{M}^* = \mathbb{P} \times \{-1\} \cup \mathbb{R} \times \{0\} \cup \mathbb{P} \times \{1\}$ with the identification $\mathbb{M} = \mathbb{R} \times \{0\}$.

Definition 1.1. Let X_α be a LOTS for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$, where γ is an ordinal. When $\gamma = 0$, we consider as $\prod_{\alpha<\gamma} X_\alpha = \{\emptyset\}$, which is a trivial LOTS, for notational conveniences. When $\gamma > 0$, every element $x \in X$ is identified with the sequence $\langle x(\alpha) : \alpha < \gamma \rangle$. Recall that the lexicographic order $<_X$ on X is defied as follows: for $x, x' \in X$,

$$x <_X x' \text{ iff for some } \alpha < \gamma,\ x \restriction \alpha = x' \restriction \alpha \text{ and } x(\alpha) < x'(\alpha),$$

where $x \restriction \alpha = \langle x(\beta) : \beta < \alpha \rangle$. Then $X = \langle X, <_X, \lambda_X \rangle$ is a LOTS and called the lexicographic product of LOTS's X_α's.

Now let X_α be a GO-space for every $\alpha < \gamma$ and $X = \prod_{\alpha<\gamma} X_\alpha$. Then the lexicographic product $\hat{X} = \prod_{\alpha<\gamma} X^*_\alpha$, which is a LOTS, can be defined. The *lexicographic product of GO-spaces* X_α's is the GO-space $\langle X, <_{\hat{X}} \restriction X, \lambda_{\hat{X}} \restriction X \rangle$. Obviously this definition extends the lexicographic product of LOTS's, and is reasonable because each X^*_α is the smallest LOTS which contains X_α as a dense subspace, see [4]. When $n \in \omega$, then $\prod_{i<n} X_i$ is denoted by $X_0 \times \cdots \times X_{n-1}$. If all X_α's are X, then $\prod_{\alpha<\gamma} X_\alpha$ is denoted by X^γ.