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Fibonacci type presentations and 3-manifolds ☆

James Howie [a], Gerald Williams [b,*]

[a] Department of Mathematics and Maxwell Institute for Mathematical Sciences, Heriot-Watt University, Edinburgh EH14 4AS, UK
[b] Department of Mathematical Sciences, University of Essex, Wivenhoe Park, Colchester, Essex CO4 3SQ, UK

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ABSTRACT

We study the cyclic presentations with relators of the form $x_i x_{i+m} x_{i+k}^{-1}$ and the groups they define. These "groups of Fibonacci type" were introduced by Johnson and Mawdesley and they generalise the Fibonacci groups $F(2,n)$ and the Sieradski groups $S(2,n)$. With the exception of two groups, we classify when these groups are fundamental groups of 3-manifolds, and it turns out that only Fibonacci, Sieradski, and cyclic groups arise. Using this classification, we completely classify the presentations that are spines of 3-manifolds, answering a question of Cavicchioli, Hegenbarth, and Repovš. When n is even the groups $F(2,n)$, $S(2,n)$ admit alternative cyclic presentations on $n/2$ generators. We show that these alternative presentations also arise as spines of 3-manifolds.

1. Introduction

Let F_n be the free group of rank $n \geq 1$ with generators $x_0, \ldots, x_{n-1}$ and let $w = w(x_0, \ldots, x_{n-1})$ be a word in F_n and let $\theta : F_n \to F_n$ be an automorphism given by $x_i \mapsto x_{i+1}$ (subscripts mod n). The presentation

$$\mathcal{G}_n(w) = \langle x_0, \ldots, x_{n-1} \mid \theta^i(w)\ (0 \leq i \leq n-1) \rangle \qquad (1)$$

is called a *cyclic presentation*, and the group $G_n(w)$ that it defines is a *cyclically presented group*. Since cyclic presentations are balanced (that is, they have an equal number of generators and relations) the questions as

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* Corresponding author.
E-mail addresses: J.Howie@hw.ac.uk (J. Howie), Gerald.Williams@essex.ac.uk (G. Williams).

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to which cyclic presentations are spines of 3-manifolds and which cyclically presented groups are fundamental groups of 3-manifolds arise and have been considered by many authors (see, for example, [5]). For instance, in [11], Dunwoody provides an algorithm to determine if a cyclic presentation is the spine of a 3-manifold. There is a large body of literature showing that certain cyclic presentations arise as spines of 3-manifolds, which hence define fundamental groups of 3-manifolds (see, for example, [6,26] and the references therein). Conversely, many theorems show that certain cyclic presentations do not define the fundamental groups of hyperbolic 3-orbifolds (in particular 3-manifolds) of finite volume (see, for example, Theorem 3.1 of any of [29,1,8,6], or see [39]). We contribute to this theory by studying one three-parameter family in detail, namely the cyclic presentations $\mathcal{G}_n(m,k) = \mathcal{G}_n(x_0x_mx_k^{-1})$ (where $n \geq 1$, $0 \leq m,k \leq n-1$) and the groups $G_n(m,k)$ they define.

The groups $G_n(m,k)$ are known as the *groups of Fibonacci type* and were introduced in [25]. Their presentations $\mathcal{G}_n(m,k)$ generalise the presentations $\mathcal{F}(2,n) = \mathcal{G}_n(1,2)$ of the *Fibonacci groups* $F(2,n)$ and the presentations $\mathcal{S}(2,n) = \mathcal{G}_n(2,1)$ of the *Sieradski groups* $S(2,n)$ of [38] and the presentations $\mathcal{H}(n,m) = \mathcal{G}_n(m,1)$ of the groups $H(n,m)$ studied in [15]. Together with the cyclic presentations $\mathcal{G}_n(x_0x_kx_l)$ studied in [8,12] they form the triangular cyclic presentations – that is, cyclic presentations where the relators have length three. The groups $G_n(m,k)$ have been studied for their algebraic properties in [1,7,15,23,25,41,42] – see [43] for a survey of such results. In this article we study geometric and topological aspects of the groups $G_n(m,k)$ and their presentations.

Our starting point is two results concerning the Fibonacci and Sieradski cases. The first is that if $n \geq 2$ is even then $\mathcal{F}(2,n)$ is the spine of a closed 3-manifold and hence $F(2,n)$ is the fundamental group of a closed 3-manifold [17,19,20,9] – see Theorem 1. The second is that if $n \geq 2$ then the presentation $\mathcal{S}(2,n)$ is the spine of a closed 3-manifold and hence $S(2,n)$ is the fundamental group of a closed 3-manifold [38,4] – see Theorem 2. Motivated by these results, in [5, Problem 6] Cavicchioli, Hegenbarth and Repovš asked which presentations $\mathcal{G}_n(m,k)$ are spines of closed 3-manifolds. Related to this is the question as to which groups $G_n(m,k)$ are fundamental groups of closed 3-manifolds. With the exception of the (known to be challenging) groups $H(9,4)$, $H(9,7)$ we answer the second question in Theorem A, and this allows us to completely answer the first question in Theorem B.

These theorems show that the only cases where spines are possible are in the cases of the Fibonacci and Sieradski presentations of Theorems 1, 2, and the only cases where the group is the fundamental group of a 3-manifold are the cases of the corresponding Fibonacci and Sieradski groups and when it is a finite cyclic group. Contained within this classification is the result (Theorem 3) that for odd $n \geq 3$ the group $F(2,n)$ is the fundamental group of a 3-manifold only when $n = 3, 5$, or 7 (in which case it is cyclic). It has previously been observed [9, page 55] that the presentation $\mathcal{F}(2,3)$ is the spine of a closed 3-manifold and the presentations $\mathcal{F}(2,5)$, $\mathcal{F}(2,7)$ are not spines of 3-manifolds and so we can conclude that $\mathcal{F}(2,n)$ is the spine of a manifold if and only if $n = 3$ or n is even, confirming the expectation expressed in [9, page 55].

An alternative presentation of the Fibonacci group $F(2,2m)$ was obtained in [27, Example 1.2], namely the presentation $\mathcal{G}_m(x_0^{-1}x_1^2x_2^{-1}x_1)$; in the same way we can obtain the alternative presentation $\mathcal{G}_m(x_0x_1^2x_2x_1^{-1})$ of the Sieradski group $S(2,2m)$. In Theorem C we prove that these alternative presentations are also spines of 3-manifolds.

2. 3-manifold groups

By a *3-manifold group* we mean the fundamental group of a (not necessarily closed, compact, or orientable) 3-manifold. A 2-dimensional subpolyhedron P of a compact, connected, 3-manifold with boundary M is called a *spine* of M if M collapses to P. A 2-dimensional subpolyhedron P of a closed, connected, 3-manifold M is called a *spine* of M if $M \backslash \mathrm{Int} B^3$ collapses to P, where B^3 is a 3-ball in M. (See, for example, [30].) We shall say that a finite presentation $\mathcal{P}$ is a *3-manifold spine* if its presentation complex $K_{\mathcal{P}}$ is a spine of a