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Henselianity in the language of rings ☆

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ABSTRACT

We consider four properties of a field K related to the existence of (definable) henselian valuations on K and on elementarily equivalent fields and study the implications between them. Surprisingly, the full pictures look very different in equicharacteristic and mixed characteristic.

1. Introduction

The study of henselian fields in the language of rings started with a work by Prestel and Ziegler ([21]) where they introduced and discussed t-henselian fields. We say that a field is *t-henselian* if it is $\mathcal{L}_{\text{ring}}$-elementarily equivalent to some *henselian* field, i.e., a field admitting a nontrivial henselian valuation. Although this does not coincide with the definition given in [21], our definition and theirs are equivalent, using the $\mathcal{L}_{\text{ring}}$-definition of the henselian topology in [19, p. 203]. Real closed fields and algebraically closed fields of positive characteristic are t-henselian but may not be henselian, e.g. $\mathbb{R}$ and $\overline{\mathbb{F}}_p$ are t-henselian but not henselian. In particular, Prestel and Ziegler showed that these are not the only examples of t-henselian fields which are not henselian. These results are strongly linked to the question of which fields

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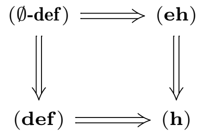

Fig. 1. The obvious implications.

admit a nontrivial definable henselian valuation. Here, we say that a valuation v is *definable* on a field K if its valuation ring $\mathcal{O}_v$ is an $\mathcal{L}_{\mathrm{ring}}$-definable subset of K (possibly with parameters from K) and that v is $\emptyset$-definable if it is definable and no parameters were needed in the defining formula. Henselianity is an elementary property of valued fields, in particular, it is preserved under elementary equivalence in the language $\mathcal{L}_{\mathrm{val}} = \mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ where the unary relation symbol $\mathcal{O}$ is interpreted as the valuation ring. Thus, if some nontrivial henselian valuation ring is a $\emptyset$-definable subring of K, then any L which is $\mathcal{L}_{\mathrm{ring}}$-elementarily equivalent to K also admits a nontrivial henselian valuation. In particular, if K is henselian and some $\mathcal{L}_{\mathrm{ring}}$-elementarily equivalent L is non-henselian, then K cannot admit a $\emptyset$-definable nontrivial henselian valuation. Under which conditions fields admit definable nontrivial henselian valuations (with or without parameters) has been investigated in a number of (mostly) recent papers [7–9,12,20] and some of these results have been applied in connection with the Shelah–Hasson conjecture on NIP fields (see [10] and [13]).

The aim of this paper is to clarify the implications and relationships between these properties of a field K, more precisely:

(**h**) K is henselian (i.e., K admits a nontrivial henselian valuation),
(**eh**) any L which is $\mathcal{L}_{\mathrm{ring}}$-elementarily equivalent to K is henselian,
(**∅-def**) K admits a $\emptyset$-definable nontrivial henselian valuation, and
(**def**) K admits a definable nontrivial henselian valuation.

We call a field *elementarily henselian* if it satisfies (**eh**). There are some immediate implications between these properties, as summarised in the diagram in Fig. 1.[1]

Our aim is to work out the full picture, i.e., to describe which other implications hold, including which arrows can be reversed. It turns out that in the class of all fields (or even in the class $\mathcal{K}_0$ of all non-algebraically closed fields of characteristic zero), no implications hold that are not already included in Fig. 1 (see part (C) of Theorem 1.1).

In order to show this, we use the canonical henselian valuation v_K to partition $\mathcal{K}_0$ into subclasses, depending on the residue characteristic of v_K:

$$\mathcal{K}_{0,0} = \{K \text{ field} \mid \mathrm{char}(K) = \mathrm{char}(Kv_K) = 0, K \text{ not algebraically closed}\}$$

and for any prime p

$$\mathcal{K}_{0,p} = \{K \text{ field} \mid \mathrm{char}(K) = 0 \text{ and } \mathrm{char}(Kv_K) = p\}.$$

See section 2 for the definition of the canonical henselian valuation and a proof that these classes are closed under $\mathcal{L}_{\mathrm{ring}}$-elementary equivalence. We then investigate the corresponding pictures with respect to these subclasses which surprisingly turn out to look rather different in mixed characteristic and equicharacteristic 0. As our main result, we obtain the following

[1] Our convention is that such diagrams implicitly include concatenations of arrows, although we do not draw them. For example, Fig. 1 implicitly includes the implication (**∅-def**) $\Longrightarrow$ (**h**).