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Modularity results for interpolation, amalgamation and superamalgamation

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ABSTRACT

Wolter in [38] proved that the Craig interpolation property transfers to fusion of normal modal logics. It is well-known [21] that for such logics Craig interpolation corresponds to an algebraic property called *superamalgamability*. In this paper, we develop model-theoretic techniques at the level of first-order theories in order to obtain general combination results transferring *quantifier-free interpolation* to unions of theories over non-disjoint signatures. Such results, once applied to equational theories sharing a common Boolean reduct, can be used to prove that superamalgamability is modular also in the non-normal case. We also state that, in this non-normal context, superamalgamability corresponds to a strong form of interpolation that we call "comprehensive interpolation property" (which consequently transfers to fusions).

1. Introduction

Craig's interpolation theorem [7] is a model-theoretic result which applies to first order formulae and states that whenever a formula ϕ entails a formula ψ, then it is possible to find a third formula θ which can be interpolated between ϕ and ψ, and which is defined over their common symbols. Interpolation theory has been recently introduced in verification, after the work of McMillan (see, e.g., [23]), and it has also a long tradition in non-classical logics (see for instance the seminal papers by L.L. Maksimova [20], [21]). In particular, the specific form of interpolation for modal logic is the following: a modal logic L is said to enjoy the (local) interpolation property iff, whenever we consider two modal formulae t_1 and t_2 such that $\vdash_L t_1 \to t_2$ holds, it is possible to find a modal formula u such that (i) $\vdash_L t_1 \to u$, (ii) $\vdash_L u \to t_2$, and (iii) the variables of u are in common with both t_1 and t_2. In this context, of great importance is the study of combination of logics, focusing on the transfer of significant properties like interpolation. The simplest

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way of combining modal logics is given by the well-known notion of *fusion*: considering two modal logics L_1 and L_2 over the modal signatures Σ^1_M and Σ^2_M such that $\Sigma^1_M \cap \Sigma^2_M = \emptyset$, the *fusion* $L_1 \oplus L_2$ is the least modal logic, over the modal signature $\Sigma^1_M \cup \Sigma^2_M$, that contains $L_1 \cup L_2$.

In [38] Wolter proved that the fusion of two interpolating *normal* modal logics is also interpolating. However, the non-normal case remained open and in this paper we try to attack it: *we show that superamalgamability transfers to fusions in the general non-normal context.* It is well-known that superamalgamability (which is an algebraic condition) is equivalent to interpolation in the normal case [21]. Thus, our result implies Wolter's result and we prove that in the general non-normal case our result gives a fusion transfer theorem for a new strong form of interpolation (covering both local and global interpolation) which we call "comprehensive interpolation property".

The above result is obtained as a *corollary of combination techniques for first-order theories*: in fact, we specialize to the modal context modular conditions of combinability that generalize various previous works. The study of the modularity property of *quantifier-free* interpolation in first-order theories was first started in [40], where the disjoint signatures convex case was solved; in [5] – the journal version of [4] – the non-convex (still disjoint) case was also thoroughly investigated. In attacking combination problems for non-disjoint signatures, we follow the model-theoretic approach successfully employed in [11], [2], [14], [29], [26], [27], [28] for combined satisfiability; this approach relies on model-theoretic notions like T_0-compatibility.

The paper is organized in five sections. In Section 2, we introduce notations and basic ingredients concerning first order logic. In Section 3 we obtain a first general result (Theorem 3.1) which gives sufficient conditions for the transfer of quantifier-free interpolation in the non-disjoint signatures case; Theorem 3.1 has all known results for disjoint signatures case [5], [40] as an immediate consequence. In Section 4 we focus our attention on universal Horn theories, in order to obtain a modular condition (Theorem 4.1) referring to "minimal" amalgams. In Section 5, we apply Theorem 4.1 to modal logic: we prove that superamalgamability is a modular condition (Corollary 5.2), since it is equivalent to the combination condition of Theorem 4.1 in case the background theory is the theory of Boolean Algebras. Then, we syntactically characterize superamalgamability, by defining the notion of "comprehensive interpolation". Comprehensive interpolation, in the normal case, is nothing but standard interpolation, but in the non-normal case it looks like a stronger property, which transfers to fusions as a consequence of our results (Theorem 5.1).

2. Formal preliminaries in first order logic

We adopt the usual first-order syntactic notions of signature, term, atom, (ground) formula, sentence, and so on. Let Σ be a first-order signature; we assume the binary equality predicate symbol '=' to be added to any signature (so, if $\Sigma = \emptyset$, then Σ just contains equality). The signature obtained from Σ by adding to it a set $\underline{a}$ of new constants (i.e., 0-ary function symbols) is denoted by $\Sigma^{\underline{a}}$. A *literal* is an atomic formula or the negation of an atomic formula; a *clause* is a disjunction of literals and a *positive clause* is a disjunction of atoms. A formula is *quantifier-free* (or open) iff it does not contain quantifiers. A Σ-*theory* T is a set of sentences (called the axioms of T) in the signature Σ and it is *universal* iff it has universal closures of open formulae as axioms.

We also assume the usual first-order notion of interpretation and truth of a formula, with the proviso that the equality predicate = is always interpreted as the identity relation. A formula φ is *satisfiable* in $\mathcal{M}$ iff its *existential* closure is true in $\mathcal{M}$. A Σ-structure $\mathcal{M}$ is a *model* of a Σ-theory T (in symbols $\mathcal{M} \models T$) iff all the sentences of T are true in $\mathcal{M}$. If φ is a formula, $T \models \varphi$ ('φ *is a logical consequence of* T') means that the universal closure of φ is true in all the models of T; T is *consistent* iff it has a model. A sentence φ is T-consistent iff $T \cup \{\varphi\}$ is consistent. A Σ-theory T is *complete* iff for every Σ-sentence φ, either φ or $\neg\varphi$ is a logical consequence of T. T admits *quantifier elimination* iff for every formula $\varphi(\underline{x})$ there is a quantifier-free