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Generic expansion and Skolemization in NSOP_1 theories

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ABSTRACT

We study expansions of NSOP_1 theories that preserve NSOP_1. We prove that if T is a model complete NSOP_1 theory eliminating the quantifier $\exists^\infty$, then the generic expansion of T by arbitrary constant, function, and relation symbols is still NSOP_1. We give a detailed analysis of the special case of the theory of the generic L-structure, the model companion of the empty theory in an arbitrary language L. Under the same hypotheses, we show that T may be generically expanded to an NSOP_1 theory with built-in Skolem functions. In order to obtain these results, we establish strengthenings of several properties of Kim-independence in NSOP_1 theories, adding instances of algebraic independence to their conclusions.

1. Introduction

Many of the early developments in the study of simple theories were guided by the thesis that a simple theory can be understood as a stable theory plus some 'random noise.' This loose intuition became a concrete recipe for creating new simple theories: start with a stable theory, and, through some kind of generic construction, add additional random structure in an expanded language. This strategy was pursued by Chatzidakis and Pillay [3], who showed that adding a generic predicate or a generic automorphism to a stable theory results in a simple theory which is, in general, unstable. In the case of adding a generic predicate, it suffices to assume that the base theory is simple; that is, expansion by a generic predicate preserves simplicity. The paper [3] spawned a substantial literature on generic structures and simple theories, which in turn shed considerable light on what a general simple theory might look like.

We are interested in using generic constructions to produce new examples of NSOP_1 theories. The class of NSOP_1 theories, which contains the class of simple theories, was isolated by Džamonja and Shelah [7] and later investigated by Shelah and Usvyatsov [13]. Until recently, very few non-simple examples were known to lie within this class. A criterion, modeled after the well-known theorem of Kim and Pillay characterizing

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the simple theories as those possessing a well-behaved independence relation, was observed by Chernikov and the second-named author in [5]. This criterion was applied to show that the theory of an ω-free PAC field of characteristic zero and the theory of an infinite dimensional vector space over an algebraically closed field with a generic bilinear form are both NSOP_1. That paper also showed, by a variation on a construction of Baudisch, that a simple theory obtained as a Fraïssé limit with no algebraicity may be 'parametrized' to produce an NSOP_1 theory which is, in general, non-simple. Later, Kaplan and the second-named author developed a general theory of independence in NSOP_1 theories, called *Kim-independence*, which turns out to satisfy many of the familiar properties of forking independence in simple theories (e.g. extension, symmetry, the independence theorem, etc.) [12]. In this paper, we apply this theory of independence to verify that certain generic constructions preserve NSOP_1.

In Section 2, we review the theory of Kim-independence in NSOP_1 theories and make some technical contributions to this theory. We establish strengthenings of the extension property, the chain condition, and the independence theorem for Kim-independence, obtaining additional instances of algebraic independence in their conclusions (see Definition 2.8, and Theorems 2.15, 2.18, and 2.21). The main deficiency of Kim-independence is the failure of base monotonicity, and this work can be viewed as an effort to circumvent that deficiency, since the instances of algebraic independence that we need would be automatic in the presence of base monotonicity (see Remarks 2.9 and 2.10).

Section 3 is dedicated to an analysis of the theory $T_L^\emptyset$ of the generic L-structure (the model completion of the empty theory in an arbitrary language L). The work in this section was motived by a preprint of Jeřábek [8]. In an early draft of [8], Jeřábek showed that $T_L^\emptyset$ is always NSOP_3, regardless of the language. He asked if this could be improved to NSOP_1 and if $T_L^\emptyset$ weakly eliminates imaginaries. We give positive answers to these questions, and we characterize Kim-independence and forking independence in this theory. In a subsequent draft of [8], Jeřábek also independently answered both questions.

But Jeřábek's first question suggested a much more general one. An L-theory T may be considered as an L'-theory for any language L' that contains L. A theorem of Winkler [14] establishes that, as an L'-theory, the theory T has a model completion $T_{L'}$, provided that T is model complete and eliminates the quantifier $\exists^\infty$. The theory $T_{L'}$ axiomatizes the generic expansion of T by the new constants, functions, and relations of L'. Using the theory developed in Section 2, we show that if T is NSOP_1, then $T_{L'}$ is as well; that is, generic expansions preserve NSOP_1.

In [14], Winkler also showed that if T is a model complete theory eliminating the quantifier $\exists^\infty$, then T has a generic Skolemization T_{Sk}. More precisely, if T is an L-theory, one may expand the language by adding a function symbol f_φ for each formula $\varphi(\overline{x}, y)$ of L. And T, together with axioms asserting that each $f_\varphi(\overline{x})$ acts as a Skolem function for $\varphi(\overline{x}, y)$, has a model companion. This result was used by Nübling in [9], who showed that one may Skolemize *algebraic* formulas in a simple theory while preserving simplicity. Nübling further observed that, in general, adding a generic Skolem function for a non-algebraic formula produces an instance of the tree property, and hence results in a non-simple theory. We show, however, that generic Skolemization preserves NSOP_1. By iterating, we show that any NSOP_1 theory eliminating the quantifier $\exists^\infty$ can be expanded to an NSOP_1 theory with built-in Skolem functions, and we also characterize Kim-independence in the expansion in terms of Kim-independence in the original theory. This result is of intrinsic interest, but it also provides a new technical tool in the study of Kim-independence in NSOP_1 theories, which, at least at its current stage of development, only makes sense when the base is a model. Preservation of NSOP_1 by generic expansion and generic Skolemization is established in Section 4.

2. NSOP_1 and independence

2.1. Preliminaries on NSOP_1

Throughout this section, we fix a complete theory T and a monster model $\mathbb{M} \models T$.