Annals of Pure and Applied Logic 169 (2018) 803–834

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Splitting into degrees with low computational strength [☆]

Rod Downey [a,b], Keng Meng Ng [a,b,*]

[a] School of Mathematics and Statistics, Victoria University of Wellington, PO Box 600, Wellington, New Zealand
[b] Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Republic of Singapore

ARTICLE INFO

Article history:
Received 28 April 2017
Received in revised form 26 March 2018
Accepted 14 April 2018
Available online 17 April 2018

MSC:
03D25

Keywords:
Degree splitting
c.e. degrees
Lowness
Superlow
Array computable

ABSTRACT

We investigate the extent to which a c.e. degree can be split into two smaller c.e. degrees which are computationally weak. In contrast to a result of Bickford and Mills that $\mathbf{0'}$ can be split into two superlow c.e. degrees, we construct a SJT-hard c.e. degree which is not the join of two superlow c.e. degrees. We also prove that every high c.e. degree is the join of two array computable c.e. degrees, and that not every high$_2$ c.e. degree can be split in this way. Finally we extend a result of Downey, Jockusch and Stob by showing that no totally ω-c.a. wtt-degree can be cupped to the complete wtt-degree.

1. Introduction

A classic result in computability theory is the result of Sacks which states that every c.e. set can be split into a pair of disjoint c.e. sets. Moreover, as Sacks also observed, these sets can be made low. Thus the low c.e. degrees generate the c.e. degrees under join.[1]

In the last fifty or so years, there have emerged a large number of lowness notions associated with c.e. (as well as other classes of) sets and degrees. Some notable examples include the array computable degrees, the superlow degrees, the K-trivial degrees, the contiguous degrees, the strongly jump traceable degrees, the totally ω-c.a. degrees and a whole infinite hierarchy introduced by Downey and Greenberg [4]. These concepts will, where necessary, be defined later in context.

☆ The first author thanks the Marsden Fund for support of this research. The second author is partially supported by the grants MOE-RG131/17 and MOE2015-T2-2-055.
* Corresponding author.
E-mail address: KMNg@ntu.edu.sg (K.M. Ng).
[1] All sets and degrees mentioned in this paper are c.e. unless otherwise stated.

https://doi.org/10.1016/j.apal.2018.04.004

It is natural to ask the extent to which the global structure of the c.e. degrees relates to these lowness classes. For example, the celebrated work of Nies and others shows that the K-trivial degrees form an ideal [12,9] in the superlow degrees, where A is superlow if $A' \equiv_{tt} \emptyset'$. So, in particular, not every c.e. degree is the join of two K-trivial degrees.

Most of the lowness concepts are concerned with how hard the set is to approximate. Thus A is called *array computable* if for each $f \leq_T A$, f has a uniformly ω-c.a. approximation. That is, there is a computable function h such that for all $f \leq_T A$, there is a computable function g with $f(x) = \lim_s g(x,s)$ where $|\{s : g(x,s+1) \neq g(x,s)\}| < h(x)$. We note here that this is not the original definition of array computability; it is a characterization due to Downey, Jockusch and Stob [7]. In fact, one can easily see that any choice of an order function will do for h, so typically we use $h(n) = n$.

Array computability is the uniform version of being totally ω-c.a., which has a similar definition, except that h can now depend on f. In other words, every function computable from A has an ω-c.a. approximation, hence the name "totally ω-c.a.". The totally ω-c.a. degrees form the first level of a hierarchy based on ordinals due to Downey and Greenberg [4]. Each superlow degree is array computable, and each array computable degree is totally ω-c.a.

Many results for low degrees do transfer to superlow ones. For instance, the low basis theorem which states that every nonempty Π^0_1 class has a low member, can be easily seen to be replaced by the superlow basis theorem (see for example, Downey and Hirschfeldt [5, Section 2.19.3]). Another example relevant to this paper is the result of Bickford and Mills [1] that $\mathbf{0'}$ is the join of two superlow c.e. degrees.

The first question that follows naturally from these results is of course: Is every c.e. degree the join of two superlow degrees? It is reasonable to believe or conjecture that this holds. The proof of Bickford and Mills uses the fact that $\mathbf{0'}$ is the top c.e. degree and can produce enough changes to allow the join of two c.e. sets to be complicated while maintaining that each set is itself close to being computable. This is fundamentally impossible if the degree we want to split is Turing incomplete. We prove the following:

Theorem 3.1. *There are c.e. degrees which cannot be split into the join of two superlow c.e. degrees.*

The reader might be tempted to guess that some assurance of computational power, such as highness, or being close to $\emptyset'$ in some sense, might still allow the given degree to be split in a similar way. We are able to show that this intuition is false. Indeed we show that the constructed degree can be extremely close to $\mathbf{0'}$. More specifically, such sets B can be *SJT-hard*, or even *ultrahigh*. This means that $\emptyset'$ is strongly jump traceable relative to B, as we will define in the relevant later section.

Nevertheless, we can recover part of the Bickford and Mills' result for $\mathbf{0'}$ for the high degrees:

Theorem 2.1. *Every high c.e. degree is the join of two array computable (and low) c.e. degrees.*

One might then be tempted to revise the conjecture to say that *every* c.e. degree is the join of two array computable degrees. Again, we show that this is false. In fact, we prove something stronger:

Theorem 4.1. *There is a high$_2$ c.e. degree* $\mathbf{a}$ *which is not the join of two totally ω-c.a. degrees.*

There are also low degrees where this is true.

The last section briefly explores the notion of "strong cupping"; cupping under the strong reducibility $\leq_{wtt}$. We know that the Turing degree of $\emptyset'$ is the join of two superlow degrees in the Turing degrees, but it is easy to show that this is not true in the wtt-degrees (see [7]). In our section, we prove something stronger. No c.e. set of totally ω-c.a. degree can even be wtt-cupped to the complete *wtt*-degree.

Theorem 5.1. *No totally ω-c.a. set can be wtt-cupped. That is, if $\emptyset' \leq_{wtt} A \oplus D$ and A is totally ω-c.a., then $\emptyset' \leq_{wtt} D$.*