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On the class of flat stable theories ☆

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ABSTRACT

A new notion of independence relation is given and associated to it, the class of flat theories, a subclass of strong stable theories including the superstable ones is introduced. More precisely, after introducing this independence relation, flat theories are defined as an appropriate version of superstability. It is shown that in a flat theory every type has finite weight and therefore flat theories are strong. Furthermore, it is shown that under reasonable conditions any type is non-orthogonal to a regular one. Concerning groups in flat theories, it is shown that type-definable groups behave like superstable ones, since they satisfy the same chain condition on definable subgroups and also admit a normal series of definable subgroup with semi-regular quotients.

1. Introduction

The notions of forking, orthogonality and regular types, among others, play a fundamental role in understanding the structure of stable theories. These were not only essential to carry out the classification programme, inside stable theories, but also have turned out to be relevant for the developments of geometric stability theory.

A stationary type is *regular* if it is orthogonal to all its forking extensions; recall that two stationary types p and q are *orthogonal* if, for any set C over which both types are based and any realizations $a \models p|C$ and $b \models q|C$, we have that $a \mathop{\smile\!\!\!\!\!\!\,|}_C b$. Minimal types are the simplest example of regular types, where forking means being algebraic. Similar to minimal ones, regular types carry a notion of geometry associated to their

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set of realizations, and hence a dimension. Their main feature is that any type can be coordinatizated by regular ones, as long as the theory contains enough regular types. Consequently, their associated geometries determine many properties of the theory.

Formally, the fact that a theory has enough regular types can be rephrased as follows: Every type is non-orthogonal to a regular one. This holds for superstable theories but this property is not exclusive of superstability. Therefore, one may try to find reasonable conditions beyond superstability which yield the existence of enough regular types. In this paper we pursue this line of investigation on an attempt to find some reasonable structure theory beyond superstability.

We introduce the class of flat theories, a subclass of stable theories which extends superstability, and analyze the existence of regular types in this context. More precisely, in Section 2 we define the notion of ω-forking,[1] which implies the usual notion of forking, and show that in a stable theory it satisfies the usual properties of independence (see Theorem 2.12), except algebraicity since it can be the trivial relation. Afterwards, a flat theory is defined as a stable theory where every type does not ω-fork over a finite set. Since non-forking implies non-ω-forking, it follows immediately that a superstable theory is flat. As in the superstable case, a notion of ordinal-valued rank among types, called U_ω-rank, is available and we point out some of its basic properties, such as the Lascar inequalities.

In the third section, a more careful analysis of flat theories is carried out. Roughly speaking, we see that any type has a non-ω-forking extension which is non-orthogonal to a regular type. Consequently, every type is close to be non-orthogonal to a regular one, see Theorem 3.9. In particular, if all forking extensions of a type are also ω-forking, then it is non-orthogonal to a regular type. This is Corollary 3.10. Nevertheless, we cannot ensure that in general every type is non-orthogonal to a regular one, but we show that flat theories are strong (Theorem 3.20) and consequently every type is non-orthogonal to a type of weight one. In fact, this holds locally for a flat type under the mere assumption that the theory is stable.

Finally, in the last section groups in flat theories are analyzed. We show that any type-definable group in a flat theory looks like a superstable one, in the sense that they satisfy the same descending chain condition on definable subgroups and also admit a semi-regular decomposition. It should be noted that, while the notion of p-semi-regularity (also p-simplicity) originated in [7, Chapter V], here semi-regularity corresponds to a reformulation due to Hrushovski. Hence, in Theorem 4.5, by a semi-regular decomposition we mean that every such flat group admits a finite series of normal subgroups such that any generic type of each quotient is domination-equivalent to suitable finite product of some regular type.

2. A new independence relation

From now on, we work inside the monster model of a complete stable first-order theory, and we assume that the reader is familiarized with the general theory of stability theory.

2.1. Skew dividing and ω-forking

We introduce the notion of skew k-dividing and k-forking for a natural number $k \geq 1$.

Definition 2.1. Let $\pi(\bar{x})$ be a partial type. It is said to skew k-divide over A if there is an A-indiscernible sequence $(\bar{b}_n)_{n<\omega}$ and a formula $\varphi(\bar{x}; \bar{y}_0, \ldots, \bar{y}_{k-1})$ such that

$$\pi(\bar{x}) \vdash \varphi(\bar{x}; \bar{b}_0, \bar{b}_2, \ldots, \bar{b}_{2(k-1)}) \text{ and } \pi(\bar{x}) \vdash \neg\varphi(\bar{x}; \bar{b}_{i_0}, \ldots, \bar{b}_{i_{k-1}})$$

for any $i_0 < \ldots < i_{k-1} < 2k$ with $(i_0, i_1, \ldots, i_{k-1}) \neq (0, 2, \ldots, 2(k-1))$.

[1] Originally, called gorking by the second author.