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Ehrenfeucht–Fraïssé games on ordinals ☆

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ABSTRACT

Two structures A and B are n-equivalent if player II has a winning strategy in the n-move Ehrenfeucht–Fraïssé game on A and B. Ordinals and m-coloured ordinals are studied up to n-equivalence for various values of m and n.

1. Introduction

Let A and B be coloured linear orders. We say that A is n-equivalent to B, written $A \equiv_n B$, if player II has a winning strategy in the n-move Ehrenfeucht–Fraïssé game on A and B. In [7] we established bounds on the least representatives of the n-equivalence classes of coloured linear orders in the special cases in which the ordering is finite, or the number of moves is at most 2. Here our focus is on the case of ordinals, with or without colours. Since the pioneering work on this by Ehrenfeucht and Fraïssé, such games have been extensively used in mathematical logic to analyze questions about the relations between different structures, n-equivalence being a finer relation than elementary equivalence, and to study decidability issues. Following on Ehrenfeucht's decidability result [3], Läuchli and Leonard also used games in their important paper [5] on the elementary theory of linear order, as did Bissell-Siders in [2] and [1].

We briefly recall the material from [7] on coloured orderings and games that we need. A coloured linear ordering is a triple $(A, <, F)$ where $(A, <)$ is a linear order and F is a mapping from A onto a set C which we think of as a set of colours. We just write A instead of $(A, <, F)$ provided that the ordering and

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colouring are clear. In the n-move Ehrenfeucht–Fraïssé game on coloured linear orders A and B (or indeed any relational structures) players I and II play alternately, I moving first. On each move I picks an element of either structure (his choice does not have to be from the same structure on every move), and II responds by choosing an element of the other structure. After n moves, I and II between them have chosen elements $x_1, x_2, \ldots, x_n$ of A, and $y_1, y_2, \ldots, y_n$ of B, and player II *wins* if the map taking x_i to y_i for each i is an isomorphism between induced substructures (that is, it preserves the ordering and colour), and player I wins otherwise. Intuitively, I is trying to demonstrate that there is some difference between the structures, while player II is trying to show that they are at least reasonably similar. We say that A and B are *n-equivalent* and write $A \equiv_n B$, if II has a winning strategy. It is easy to see that $\equiv_n$ is an equivalence relation, and it is standard that for any n, there are only finitely many n-equivalence classes, so it is natural to enquire what their optimal representatives may be. The problem for general orderings seems to be quite hard, but with special conditions on the type of ordering or colouring, or the number of moves, some results can be obtained. If the ordering is an ordinal, then the notion of 'optimality' makes sense: a (coloured) ordinal is *optimal* if it is least in its n-equivalence class. This may still not be unique in the coloured case. If the ordering is finite, then we may take the lexicographically least; in the general case we would hope to make some canonical choice, for instance, exhibiting some eventual periodicity.

Already in [8], some information about the optimal representatives of n-equivalence classes of (monochromatic) ordinals is given (also see [4]). Rosenstein remarks (as an exercise) that every ordinal is $2n$-equivalent to some ordinal in the finite set

$$\{\omega^n \cdot a_n + \omega^{n-1} \cdot a_{n-1} + \ldots + \omega \cdot a_1 + a_0 : a_i < 2^{2n}, a_n \leq 1\}.$$

In section 2 we sharpen this result to give precise lists of all the optimal values for n-equivalence classes of ordinals, including the case where n is odd.

In section 3 we move on to consider the coloured case. By [7], we already understand the situation for 2 moves, and we now generalize this to more moves. Here we concentrate on giving some upper bounds for the optimal representatives, which certainly seem unnecessarily large, but at least all lie below the ordinal ω^ω.

Next we recall the notion of 'character' from [7], and the main result about characters. Assume that we have found representatives for the n-equivalence classes of certain m-coloured linearly ordered sets. We write the representative for A as $[A]_n$. In a coloured linear order A, the *n-character* of $a \in A$ having colour c is the ordered pair $\langle [A^{<a}]_n, [A^{>a}]_n \rangle$ (where $A^{<a} = \{x \in A : x < a\}$ and $A^{>a} = \{x \in A : x > a\}$). We let $\rho_n^c(A) = \{\langle [A^{<a}]_n, [A^{>a}]_n \rangle : a \in A \text{ is } c\text{-coloured}\}$, and if we wish to include the colour as part of the n-character of a, we may also write $\langle [A^{<a}]_n, [A^{>a}]_n \rangle_c$.

Theorem 1.1 *([7]). $A \equiv_{n+1} B$ if and only if $\rho_n^c(A) = \rho_n^c(B)$ for all $c \in C$.*

If A and B are coloured linear orders, then $A + B$ stands for the concatenation of A and B, that is, we first assume (by replacing by copies if necessary) that A and B are disjoint, and we place all members of A to the left of all members of B. As a generalization of this, we may write $\sum\{A_i : i \in I\}$ for the concatenation of a family of (coloured) linear orders $\{A_i : i \in I\}$ indexed by a linear ordering I. When forming concatenations we would normally assume that all the orderings have the same colour set. We write $A \cdot B$ for the anti-lexicographic product, B 'copies of' A, to accord with the customary use for ordinals (and unlike [7], where *lexicographic* products are used). Note that here B is assumed monochromatic, and colours are assigned to members of $A \cdot B$ by means of the A-co-ordinate. The following result will be used without explicit reference.

Theorem 1.2. (i) *If $A \equiv_n B$, then $X + A + Y \equiv_n X + B + Y$ and $X \cdot A \cdot Y \equiv_n X \cdot B \cdot Y$.*

(ii) *If $A_i \equiv_n B_i$ for each $i \in I$, then $\sum\{A_i : i \in I\} \equiv_n \sum\{B_i : i \in I\}$.*

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