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A multiplication operation for the hierarchy of norms ☆

Alexander C. Block [a], Benedikt Löwe [a,b,c,*]

[a] Fachbereich Mathematik, Universität Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany
[b] Institute for Logic, Language and Computation, Universiteit van Amsterdam, Postbus 94242, 1090 GE Amsterdam, The Netherlands
[c] Churchill College & Faculty of Mathematics, University of Cambridge, Storey's Way, Cambridge CB3 0DS, United Kingdom

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ABSTRACT

Assuming $\mathsf{AD}+\mathsf{DC}$, the hierarchy of norms is a wellordered structure of equivalence classes of ordinal-valued maps. We define operations on the hierarchy of norms, in particular an operation that dominates multiplication as an operation on the ranks of norms, and use these operations to establish a considerably improved lower bound for the length of the hierarchy of norms.

1. Introduction

As usual in set theory, we refer to the elements of Baire space ω^ω as *real numbers* and use the notation $\mathbb{R}$ for ω^ω. A surjective function φ from $\mathbb{R}$ onto some ordinal α is called a *norm*. In analogy to the usual Wadge ordering of sets of reals, we can order the norms by setting $\varphi \leq_{\mathrm{NW}} \psi$ if and only if there is a continuous

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* Corresponding author.
E-mail addresses: alexander.block@uni-hamburg.de (A.C. Block), b.loewe@uva.nl (B. Löwe).

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$f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $\varphi(x) \leq \psi(f(x))$. The First Periodicity Theorem [1] shows that in the theory $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$, this ordering is a prewellordering (cf. Theorem 3).

As a consequence, in $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$, we can define an ordinal Σ, the *length of the hierarchy of norms*, to be the order-type of $\leq_{\mathrm{NW}}$. Defining as usual $\Theta := \sup\{\alpha;$ there is a surjection from $\mathbb{R}$ to $\alpha\}$, the second author proved in [6, Corollary 3 and Theorem 5] that $\Theta^2 \leq \Sigma < \Theta^+$.

In this paper, we shall improve the lower bound to $\Theta^{\left(\Theta^\Theta\right)}$. In § 2, we give the basic definitions needed for this paper. The structure theory of the Wadge hierarchy will serve as a template for the later sections; in § 3, we give a brief overview of this structure theory. One particularly important notion for the Wadge hierarchy is the notion of self-duality; in § 4, we introduce the analogous notion for the hierarchy of norms and some basic operations that are specific for the hierarchy of norms. In § 5, we discuss operations on the hierarchy of norms that can be directly transferred from the Wadge hierarchy (such as addition).

The heart of the paper are §§ 6 & 7, where we define an operation for the hierarchy of norms dominating multiplication on the ranks of norms (which is the analogue of the multiplication operation for the Wadge hierarchy defined by Steel in [7, § III.D]) and prove the main theorem about it (Theorem 22). Finally, in § 8, we apply this theorem to get the improved lower bound for Σ (Theorem 28) and finish with some open questions.

2. Basic definitions & properties

Our axiomatic framework will be ZF; if we assume additional axioms, we shall state them explicitly. Our main object of study will be the set of functions from the reals to Θ, i.e., $\Theta^{\mathbb{R}}$. We write

$$\mathrm{lh}(\varphi) := \sup\{\alpha + 1\,;\, \alpha \in \varphi[\mathbb{R}]\}$$

for the *length* of a function $\varphi : \mathbb{R} \to \Theta$. A function φ is called a *weak norm* if $\mathrm{lh}(\varphi) < \Theta$. We have the following fact:

Lemma 1. *The ordinal Θ is singular if and only if there is some $\varphi : \mathbb{R} \to \Theta$ with $\mathrm{lh}(\varphi) = \Theta$; thus, Θ is regular if and only if the set of weak norms coincides with $\Theta^{\mathbb{R}}$.*

Proof. If $\mathrm{cf}(\Theta) = \alpha < \Theta$, then there are a cofinal $f : \alpha \to \Theta$ and a surjection $g : \mathbb{R} \to \alpha$, thus $\mathrm{lh}(f \circ g) = \Theta$. Conversely, if $\varphi : \mathbb{R} \to \Theta$, then $\alpha := \mathrm{ot}(\mathrm{ran}(\varphi)) < \Theta$ by definition of Θ. For $\beta < \alpha$, we define $f(\beta)$ to be the βth element of $\mathrm{ran}(\varphi)$. If $\mathrm{lh}(\varphi) = \Theta$, then $f : \alpha \to \Theta$ is cofinal, and hence $\mathrm{cf}(\Theta) \leq \alpha < \Theta$. □

A weak norm is called a *norm* if its range is an ordinal. If φ is a norm, then $\mathrm{lh}(\varphi) = \mathrm{ran}(\varphi)$. We denote the set of weak norms by $\mathrm{w}\mathcal{N}$ and the set of norms by $\mathcal{N}$.

A relation is called a *preorder* if it is transitive and reflexive. If $\leq$ is a preorder on a set X, then we can define the *corresponding equivalence relation* $\equiv$ by $a \equiv b :\Leftrightarrow a \leq b \wedge b \leq a$ for all $a, b \in X$, and the *corresponding strict preorder relation* $<$ by $a < b :\Leftrightarrow a \leq b \wedge \neg a \equiv b$ for all $a, b \in X$. A preorder $\leq$ induces a partial order on the $\equiv$-equivalence classes; we denote this partial order with the same symbol $\leq$.

As mentioned before, for $\varphi, \psi \in \Theta^{\mathbb{R}}$, we write $\varphi \leq_{\mathrm{NW}} \psi$ if and only of there is a continuous $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $\varphi(x) \leq \psi(f(x))$. We write $\varphi \leq_{\mathrm{NL}} \psi$ if there is a Lipschitz function with the same property. These relations are preorders and we denote the corresponding equivalence relations by $\equiv_{\mathrm{NW}}$ and $\equiv_{\mathrm{NL}}$ and their corresponding strict preorder relations by $<_{\mathrm{NW}}$ and $<_{\mathrm{NL}}$.

It is easy to see that if $\mathrm{lh}(\varphi) < \mathrm{lh}(\psi)$, then $\varphi <_{\mathrm{NL}} \psi$; furthermore, any two norms of length $\alpha + 1$ for some α are Lipschitz-equivalent. As usual, for $x, y \in \mathbb{R}$, we define a real $x * y$, constructed by interleaving x and y, as follows: