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The Kierstead's Conjecture and limitwise monotonic functions

Guohua Wu [a,*,1], Maxim Zubkov [a,b,2]

[a] *Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Singapore*
[b] *N.I. Lobachevsky Institute of Mathematics and Mechanics, Kazan Federal University, Kazan, Kremlevskaya 18, 420008, Russia*

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ABSTRACT

In this paper, we prove Kierstead's conjecture for linear orders whose order types are $\sum_{q\in\mathbb{Q}} F(q)$, where F is an extended $0'$-limitwise monotonic function, i.e. F can take value ζ. Linear orders in our consideration can have finite and infinite blocks simultaneously, and in this sense our result subsumes a recent result of C. Harris, K. Lee and S.B. Cooper, where only those linear orders with finite blocks are considered. Our result also covers one case of R. Downey and M. Moses' work, i.e. $\zeta \cdot \eta$. It covers some instances not being considered in both previous works mentioned above, such as $m \cdot \eta + \zeta \cdot \eta + n \cdot \eta$, for example, where $m, n > 0$.

1. Introduction

In this paper, we make progress toward solving a long-standing open problem of characterizing the order types of Π^0_1-rigid computable linear orders (a linear order is Π^0_1-rigid if it does not have any nontrivial Π^0_1-automorphism). Downey's survey paper [1] provides an extensive motivation and research toward this problem. As usual, ω, ζ, η are the order types of natural, integers and rational numbers, respectively. Moreover, $\mathbb{N}$, $\mathbb{Q}$ are the set of natural numbers and the set of rational numbers, correspondingly. By technical reasons, we assume that $\mathbb{N}$ does not contain 0.

* Corresponding author.
E-mail addresses: guohua@ntu.edu.sg (G. Wu), maxim.zubkov@kpfu.ru (M. Zubkov).
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L. Hay and J. Rosenstein proved that the well-known Dushnik–Miller theorem [4] saying that an infinite countable linear order has a nontrivial self-embedding is not effective.

Theorem 1 *(L. Hay, J. Rosenstein (in [10])). There is a computable copy of ω with no nontrivial computable self-embedding.*

By using a standard back-and-forth argument, it is easy to see that if a linear order $\mathcal{L}$ has order type η then every computable copy of $\mathcal{L}$ has a nontrivial computable automorphism. S. Schwarz gave a characterization of linear orders with nontrivial computable automorphisms.

Theorem 2 *(S. Schwarz [11,12]). Every computable linear order $\mathcal{L}$ has a computably rigid computable copy if and only if it contains no interval of order type η. Here a linear order is computably rigid if it has no nontrivial computable automorphism.*

The investigation for η-like linear orders was initiated by H. Kierstead in his paper [9]. Recall that a linear order $\mathcal{L}$ is η-like if $\mathcal{L}$ is isomorphic to $\sum_{q \in \mathbb{Q}} F(q)$ for some function $F : \mathbb{Q} \to \mathbb{N}$, and we say that the order type of $\mathcal{L}$ is defined by function F. Kierstead considered $2 \cdot \eta$, a particular instance of η-like computable linear orders, where he used an infinite injury argument to construct a computable copy of $2 \cdot \eta$ with no nontrivial Π^0_1 automorphism.

Theorem 3 *(H. Kierstead [9]). There is a computable linear order of order type $2 \cdot \eta$ which has no nontrivial Π^0_1 automorphism.*

According to H. Kierstead in paper [9], an automorphism f is fairly trivial if for all $x \in L$, there are only finitely many elements between x and $f(x)$. A nontrivial automorphism is called strongly nontrivial, if it is not fairly trivial. H. Kierstead's paper [9] concluded with 3 conjectures, with the main one as follows.

Conjecture *(H. Kierstead [9]). Every computable copy of a linear order $\mathcal{L}$ has a strongly nontrivial Π^0_1 automorphism if and only if $\mathcal{L}$ contains an interval of order type η.*

By Theorem 3, this conjecture is true for order type $2 \cdot \eta$. Later, R. Downey and M. Moses proved that H. Kierstead's conjecture also holds for discrete linear orders, where a linear order is discrete if every element has both an immediate predecessor and an immediate successor except for the possible first and last elements.

Theorem 4 *(R. Downey, M. Moses [3]). Every computable discrete linear ordering has a computable copy with no strongly nontrivial Π^0_1 self-embedding.*

A recent work of C. Harris, K. Lee and S.B. Cooper [7] extended H. Kierstead's result, where they proved that Kierstead's conjecture is true for a quite general subclass of η-like computable linear orders.

Definition 1. A function F is called *X-limitwise monotonic*, if there is an X-computable function $f(x, s)$ such that

1. $(\forall x)(\forall s)[f(x, s) \leq f(x, s+1)]$;
2. $(\forall x)[F(x) = \lim_{s \to \infty} f(x, s)]$.

C. Harris, K. Lee and S.B. Cooper [7] proved that every linear order with no dense intervals whose order type is defined by a $\mathbf{0}'$-l.m.f. has a Π^0_1-rigid computable copy.