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The tree property at the double successor of a singular cardinal with a larger gap

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ABSTRACT

Starting from a Laver-indestructible supercompact κ and a weakly compact λ above κ, we show there is a forcing extension where κ is a strong limit singular cardinal with cofinality ω, $2^\kappa = \kappa^{+3} = \lambda^+$, and the tree property holds at $\kappa^{++} = \lambda$. Next we generalize this result to an arbitrary cardinal μ such that $\kappa < \mathrm{cf}(\mu)$ and $\lambda^+ \leq \mu$. This result provides more information about possible relationships between the tree property and the continuum function.

1. Introduction

In [2], Cummings and Foreman showed that starting from a Laver-indestructible supercompact cardinal κ and a weakly compact $\lambda > \kappa$, one can construct a generic extension where $2^\kappa = \lambda = \kappa^{++}$, κ is a singular strong limit cardinal with cofinality ω, and the tree property holds at κ^{++}. It is natural to try to generalize this result in at least two directions.

First, one can ask whether – in addition to the properties identified in the previous paragraph – κ can equal $\aleph_\omega$. Cummings and Foreman suggested in [2] that this is possible, but did not provide any details. A model with the tree property at $\aleph_{\omega+2}$, with $\aleph_\omega$ strong limit, was first constructed by Friedman and Halilović in [3], moreover from a significantly lower large cardinal assumption of hypermeasurability.[2] Shortly afterwards, Gitik, answering a question posed in [3], showed in [7] that the same result can be proved from a weaker and optimal assumption.

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[2] The technique of proof in [3] used the Sacks forcing to obtain the tree property, unlike the proof in [2] which is based on a Mitchell-style analysis.

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Second, one can ask whether it is possible to have 2^κ greater than κ^{++} with the tree property at κ^{++}. Using a variant of the Mitchell forcing, Friedman and Halilović [4] proved that starting from a sufficiently hypermeasurable κ, one can keep the measurability of κ together with $2^\kappa > \kappa^{++}$ and the tree property at κ^{++}.

In this paper, we generalize [2] in the second direction. In Theorem 3.1, we prove that starting from a Laver-indestructible supercompact κ and a weakly compact λ above, one can find a forcing extension where κ is a strong limit singular cardinal with cofinality ω, $2^\kappa = \kappa^{+3} = \lambda^+$, and the tree property holds at κ^{++}. In Theorem 4.1 we give an outline of a generalization in which the gap $(\kappa, 2^\kappa)$ can be arbitrarily large: $2^\kappa = \mu$ for any cardinal $\mu > \lambda$ with cofinality greater than κ. The method of the proof is in general based on the argument of Cummings and Foreman [2], with the final part of the argument following Unger's presentation in [10].[3]

The basic idea of our proof is as follows: Recall that the basic Mitchell forcing for obtaining $2^\kappa = \kappa^{++} = \lambda$ with the tree property at λ, as presented for instance in Abraham [1], can be viewed as being composed of two components: of the Cohen forcing for adding λ-many subsets of κ (we denote this forcing by $\mathrm{Add}(\kappa, \lambda)$), and of the collapsing component which ensures $2^\kappa = \kappa^{++} = \lambda$ in the final extension. It is important that the collapsing component at stage $\alpha < \lambda$ depends on the first α-many subsets of κ added by the Cohen forcing, i.e. on $\mathrm{Add}(\kappa, \alpha)$. Cummings and Foreman generalized this idea by making the first component of the Mitchell forcing more complex: they made the collapsing part at stage $\alpha < \lambda$ depend not only on $\mathrm{Add}(\kappa, \alpha)$, but on $\mathrm{Add}(\kappa, \alpha)$ followed by the Prikry forcing on κ defined with respect to a certain normal measure U_α existing in the generic extension by $\mathrm{Add}(\kappa, \alpha)$. To make this definition coherent, these U_α's are obtained uniformly from a single measure U which exists in the extension of V by $\mathrm{Add}(\kappa, \lambda)$. Importantly, they still retain the same length of the first component (now a Cohen forcing followed by a Prikry forcing) and the collapsing component (they both have length λ).

In our case, we would like to add μ-many subsets of κ, with the final measure U living in the extension by $\mathrm{Add}(\kappa, \mu)$, where μ is typically much larger than λ. This introduces a mismatch between the length of the first and second component of the Mitchell forcing (our collapsing component needs to have the same length as before, i.e. λ). We solve this problem by reflecting U more carefully and in several stages. To simplify the exposition, we first provide the argument for the special case of $\mu = \lambda^+$ (Theorem 3.1), and articulate the modifications for the general case only later (Theorem 4.1).

The argument in Theorem 3.1 (and implicitly also the argument in Theorem 4.1) is divided into two stages. In the first stage, Section 3.1, we fix some β_0, $\lambda < \beta_0 < \lambda^+$ (Theorem 3.1), or in general some $y_0 \in [\mu]^\lambda$, $\lambda + 1 \subseteq y_0$ (Theorem 4.1), which reflects the measure U on a set of size λ, and we also fix the associated bijection π between β_0 (or y_0) and the even ordinals below λ.[4] These fixed objects are used to define the main forcing $\mathbb{R}$ while ensuring that the measure U – or more precisely its π-image – becomes a normal measure in the extension of V by the Cohen forcing defined on cofinally many even coordinates below λ.[5] The definition of the collapsing component of $\mathbb{R}$ uses only the even ordinals below λ for the reason of reserving some free space for conditions in the forcing $\mathrm{Add}(\kappa, \mu)$ which do not have a role in the collapsing component of $\mathbb{R}$, but need to be mapped onto the odd ordinals below λ by the following argument: Assuming for contradiction that $\mathbb{R}$ adds a λ-Aronszajn tree $\dot{T}$, we choose some β^* (which again reflects U), $\beta_0 < \beta^* < \lambda^+$, or in general $y^* \in [\mu]^\lambda$, $y_0 \subsetneq y^*$, which is large enough to contain all coordinates in $\mathrm{Add}(\kappa, \mu)$ which appear in $\dot{T}$.[6] We argue that $\mathbb{R}$ naturally restricts to β^*, or y^*, and moreover is isomorphic to a

[3] There seems to be a problem with Lemma 7.1 in [2] which states that a certain forcing is κ^+-Knaster, but without a convincing proof. Unger in [10] proved a version of Lemma 7.1, weakening κ^+-Knasterness to "κ^+-square-cc", which is still sufficient to conclude the whole proof. See Lemmas 3.22 and 3.28 in the present paper which follow Unger's presentation.

[4] β_0 is fixed in (3.2) and y_0 in item (2) of the proof of Theorem 4.1.

[5] Definition 3.9 for Theorem 3.1 and item (4) for Theorem 4.1.

[6] β^* introduced below Lemma 3.13, and y^* is fixed in item (5).