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Density of the cototal enumeration degrees

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A R T I C L E I N F O

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We prove that the cototal enumeration degrees are exactly the enumeration degrees of sets with good approximations, as introduced by Lachlan and Shore [17]. Good approximations have been used as a tool to prove density results in the enumerations degrees, and indeed, we prove that the cototal enumerations degrees are dense.

1. Introduction

Enumeration reducibility captures a natural relationship between sets of natural numbers in which positive information about the first set is used to produce positive information about the second set. It has a complicated history, being intimately connected to the idea of computing with partial oracles. Equivalent forms of this reducibility have been introduced several times over the years, by Kleene [16], Myhill [21], and Selman [22], however Friedberg and Rogers [7] provided us with the most convenient way of thinking about it. A set A is *enumeration reducible* to a set B if there is a uniform way to enumerate A given any enumeration of B.

Definition 1.1. $A \leq_e B$ if and only if there is a c.e. set W such that

$$x \in A \Leftrightarrow (\exists v)[\langle x, v\rangle \in W \;\&\; D_v \subseteq B].$$

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Here D_v denotes the finite set with code v in the standard coding of finite sets. The set W is called an *enumeration operator* and the pair $\langle x, v\rangle$ is called an *axiom* for x in W.

Enumeration reducibility induces a partial order, $\mathcal{D}_e$, called the enumeration degrees. This partial order attracts interest for, among other reasons, the fact that it contains a substructure that is isomorphic to the structure of the Turing degrees: the total enumeration degrees.

Definition 1.2. A set A is *total* if $\overline{A} \leq_e A$. An enumeration degree is *total* if it contains a total set.

The structure of the Turing degrees, $\mathcal{D}_T$, properly embeds into $\mathcal{D}_e$, i.e., there are enumeration degrees that are not total. Once this was established, the questions that drove the study of the enumeration degrees were mostly aimed at understanding to what extent the two degree structures are alike and in which respects they differ. One question, in particular, that attracted a lot of interest was whether $\mathcal{D}_e$ has minimal degrees: a degree is minimal if it is not the least degree, $\mathbf{0}_e$, but the only degree strictly below it is $\mathbf{0}_e$. In 1971, Gutteridge [12] proved that the enumeration degrees are downwards dense, and hence there are no minimal degrees in $\mathcal{D}_e$. On the other hand, Cooper [6] showed that there are empty intervals of enumeration degrees. In 1984, Cooper [5] showed that the Σ^0_2 enumeration degrees are a dense substructure of $\mathcal{D}_e$. In view of this result, he posed a challenge: identify the least level of the arithmetical hierarchy containing a set whose enumeration degree is the top of an empty interval, i.e., a minimal cover. To give a further incentive for his challenge, he added a conjecture: he believed that the Π^0_2 enumeration degrees are dense. Lachlan and Shore [17] extended Cooper's density result and at the same time significantly limited the possibilities for the existence of minimal covers: they introduced the notion of a *good approximation* and proved that degrees of sets with good approximations, *good enumeration degrees* for short, cannot be or have minimal covers. They gave two examples of classes of good enumeration degrees: the total enumeration degrees and the enumeration degrees of the n-c.e.a. sets for every natural number n. On the other hand, they proved that there is a Π^0_2 set that does not have a good approximation, suggesting that Cooper's conjecture could turn out to be false. Finally, Calhoun and Slaman [3] resolved the question by constructing a Π^0_2 set whose enumeration degree is a minimal cover.

As a byproduct of this line of research, we obtained an important technical tool, the notion of a good approximation, and a big mystery—what is a natural characterization of the class of good enumeration degrees? The two known examples of classes of good degrees, the total enumeration degrees and the n-c.e.a. enumeration degrees, can be combined to show that the enumeration degrees of sets that are n-c.e.a. relative to any total oracle are good enumeration degrees. But does this exhaust the class of good enumeration degrees? On the other hand, good approximations were introduced as a way to extend Cooper's density result. Lachlan and Shore [17] do in fact obtain a density result using their tool: they show the density of the enumeration degrees of n-c.e.a. sets for every fixed n. A natural questions is, therefore, whether good approximations capture a class of enumeration degrees that is itself dense. Yet, without the right characterization of the good degrees, these questions did not seem approachable. It is not even immediately obvious that the good enumeration degrees are closed under join, i.e., that they form a substructure of $\mathcal{D}_e$. Nevertheless, structural properties of good enumeration degrees have been investigated. Cooper [5] introduced an enumeration jump operator that maps an enumeration degree $\mathbf{a}$ to a strictly larger total enumeration degree $\mathbf{a}'$. Griffiths [11] showed a jump inversion theorem involving the good enumeration degrees: if $\mathbf{w}$ is a good enumeration degree and $\mathbf{x}$ is an enumeration degree such that $\mathbf{x} < \mathbf{w} \leq \mathbf{x}'$, then there is a degree $\mathbf{a}$ such that $\mathbf{x} \leq \mathbf{a} < \mathbf{w}$ and $\mathbf{a}' = \mathbf{w}'$. This result was further refined by Harris [13].

We will give several useful characterizations of the good enumeration degrees by showing that they coincide with the cototal degrees, a class that has recently been shown to have many natural properties.

Definition 1.3. A set A is *cototal* if $A \leq_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.