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# No finite axiomatizations for posets embeddable into distributive lattices

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Let $m$ and $n$ be cardinals with $3 \leq m, n \leq \omega$. We show that the class of posets that can be embedded into a distributive lattice via a map preserving all existing meets and joins with cardinalities strictly less than $m$ and $n$ respectively cannot be finitely axiomatized.



## 1. Introduction

Let $m$ and $n$ be cardinals with $3 \leq m, n \leq \omega$. It is shown in [8] that the problem of deciding whether a given finite poset can be embedded into a distributive lattice via a map preserving existing meets and joins with cardinalities strictly less than $m$ and $n$ respectively is **NP**-complete for all $m$ and $n$ except, possibly, the case where both $m$ and $n$ are equal to 3. By [9, Proposition 3.1], polynomial time algorithms exist for checking whether a fixed first-order sentence holds in finite models. So, if a class of posets with this kind of embedding property for some suitable $m$ and $n$ were finitely axiomatizable, it would imply that $\mathbf{P} = \mathbf{NP}$. Needless to say, this implication strongly suggests that none of these classes is finitely axiomatizable. However, intuitive finite first-order axiomatizations do exist for semilattices in similar situations [1,7].

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Assuming finiteness, or a suitable choice principle, this problem of embedding posets into distributive lattices is equivalent to the problem of embedding posets into powerset algebras via maps preserving meets and joins smaller than specified cardinals $m$ and $n$. Note that, since $m$ and $n$ are greater than 2, such an embedding will automatically preserve any relative complements that exist in the poset. This has been studied in [4,3] using the terminology $(m, n)$-*representable* (see Definition 2.1). In particular, it was shown that all the classes where $m, n \leq \omega$ are elementary [3, Theorem 4.5], though explicit axioms are not known. In the cases where either $m$ or $n$ is equal to $\omega$, the corresponding class is not finitely axiomatizable. This was shown directly in [3], and also follows from the corresponding result for semilattices [6]. However, the cases where $m$ and $n$ are both finite were left open.

Since for $3 \leq m, n \leq \omega$ the classes of $(m, n)$-representable posets are all elementary, they will be finitely axiomatizable if and only if their complements are elementary. By Łoś' theorem, these complements will be elementary only if they are closed under ultraproducts.

For a poset $P$ and cardinals $\alpha$ and $\beta$, the existence of an $(\alpha, \beta)$-representation for $P$ is equivalent to a separation property generalizing the separation of distributive lattices by prime filters (the Prime Ideal Theorem for distributive lattices). In this note we use this property to construct a sequence of finite posets, all of which fail to be $(3, 3)$-representable, and an ultraproduct of this sequence which is $(\omega, \omega)$-representable, thus proving that the class of $(m, n)$-representable posets cannot be finitely axiomatizable for any choice of $n, m \geq 3$.

The classes of $(\alpha, \beta)$-representable posets, when $\alpha$ and/or $\beta$ are uncountable, and the classes where *all* meets and/or joins must be preserved, are known to not be elementary at all, though in some cases they are pseudoelementary. See [4, Figure 2] for a summary.

In Section 2 we introduce the basic notation, definitions and results for representable posets (using the notation of [3]). Finally in Section 3 we construct the required sequence of posets and prove the necessary results to support our main claim.

## 2. Representable posets

We begin with some notational conventions. Given a poset $P$ and a subset $S \subseteq P$ we define $S^{\uparrow} = \{p \in P : p \geq q \text{ for some } q \in S\}$. Given $p \in P$ we define $p^{\uparrow} = \{p\}^{\uparrow}$. Given a set $I$, an ultrafilter $U$ of $\wp(I)$, and posets $P_i$ for $i \in I$ we let $\prod_U P_i$ be the ultraproduct with respect to $U$. For an element of $\prod_U P_i$ we write, e.g. $[x] \in \prod_U P_i$.

**Definition 2.1** *($(\alpha, \beta)$-representable).* Let $\alpha$ and $\beta$ be cardinals. We say a poset $P$ is $(\alpha, \beta)$-representable if there is a field of sets $F$, and a 1-1 map $h : P \to F$ such that:

1. Whenever $S$ is a subset of $P$ with $|S| < \alpha$, if $\bigwedge S$ exists in $P$, then $h(\bigwedge S) = \bigcap h[S]$.
2. Whenever $T$ is a subset of $P$ with $|T| < \beta$, if $\bigvee T$ exists in $P$ then $h(\bigvee T) = \bigcup h[T]$.

If $\alpha = \beta$ we just write $\alpha$-*representable*.

**Definition 2.2** *($(\alpha, \beta)$-filter).* Let $\alpha$ and $\beta$ be cardinals, let $P$ be a poset, and let $\Gamma$ be an up-closed subset of $P$. We say $\Gamma$ is an $(\alpha, \beta)$-filter if:

1. Whenever $S \subseteq \Gamma$ and $|S| < \alpha$, if $\bigwedge S$ exists, then $\bigwedge S \in \Gamma$.
2. Whenever $T \subseteq P$ with $|T| < \beta$, if $\bigvee T$ exists and $\bigvee T \in \Gamma$, then $T \cap \Gamma \neq \emptyset$.

I.e. $\Gamma$ is both $\alpha$-complete and $\beta$-prime. If $\alpha = \beta$ we just write $\alpha$-*filter*.

The following result relates $(\alpha, \beta)$-representability to separation by $(\alpha, \beta)$-filters. It appears explicitly in this form as [3, Theorem 2.7], but the idea of using this kind of separation property for representability-like